We propose a generalization of the rational expectations (RE) in Muth (1961).

In our framework, as in the original rational expectation approach, the case of multiple solutions is the natural case.

Expectations are then formed by randomizing across the infinite RE solutions. We call hence our approach: "rational sunspots''.

The infinite solutions differ in the way agents form their expectations, or more precisely in the way agents weight past data to calculate their expectations.
Our approach naturally yields drifting parameters and stochastic volatility.

Our framework allow for both indeterminacy (multiple stable solutions) and the possibility of (temporary?) explosive paths.

Simple method to identify sunspots vs. determinate equilibrium based on the Normality of the likelihood.

At the very least, our approach can be used as a test for RE in the sense of determinate equilibrium, that is, in choosing the unique stable equilibrium among infinitely many explosive ones.
MOTIVATION

CPI inflation, quarterly data. Sample: 1955Q1 - 2006Q4

CGG (2000) => pre Volcker: $\phi = 0.68$, Volcker-Greenspan: $\phi = 2.14$
Rational Expectations Equilibria
Weaknesses of CGG’s interpretation:

- Not so plausible: have you ever try to explain this to an engineer? theory vs. practice
- Indeterminacy means an infinite numbers of stable solutions. Why inflation seems to explode?
- Imposing $|\phi| > 1$ you gain determinacy but you may have problems in identifying parameters (Cochrane 2010)

**Question:** is there any evidence that inflation is described (at least for a while) by unstable equilibria?
A simple example

\[ y_t = \frac{1}{\lambda} E_t y_{t+1} + \omega_t \quad \omega_t \sim i.i.d. N(0, \sigma^2_\omega) \]  

(1)

\[ \lambda > 1 \implies \text{eigenvalues outside the unit circle} \implies \text{stable in forward dynamics} \]
\[ \lambda < 1 \implies \text{eigenvalues inside the unit circle} \implies \text{stable in backward dynamics} \]
Muth (1961) $\Rightarrow$ all the solutions for $y_t$ are described by

$$y_t = \sum_{j=1}^{\infty} u_j \omega_{t-j} + b \omega_t + \sum_{j=1}^{\infty} c_j E_t \omega_{t+j}$$  \hspace{1cm} (2)

$$u_1 = \lambda (b - 1) \hspace{1cm} u_{j+1} = \lambda u_j \hspace{0.5cm} j = 1, 2, \ldots, \infty$$  \hspace{1cm} (3)

$$c_1 = \frac{b}{\lambda} \hspace{1cm} c_{j+1} = \frac{c_j}{\lambda} \hspace{0.5cm} j = 1, 2, \ldots, \infty$$  \hspace{1cm} (4)

- Equation (2) is a solution for model (1) if the above conditions about coefficients hold.
- Degree of freedom: $b \in (-\infty, +\infty)$
A simple example

Two particular cases:

1. \( b = 1 \): pure forward-looking solution

\[
y^F_t = \sum_{j=0}^{\infty} \left( \frac{1}{\lambda} \right)^j E_t \omega_{t+j} = \omega_t
\]  

(5)

2. \( b = 0 \): pure backward-looking solution

\[
y^B_t = - \sum_{j=1}^{\infty} \lambda^j \omega_{t-j} = -\lambda \omega_{t-1} - \lambda \sum_{j=1}^{\infty} \lambda^j \omega_{t-j-1} =
\]

\[
= \lambda \left( y^B_{t-1} - y^F_{t-1} \right)
\]  

(6)

All the solutions can be written as a linear combination of the forward and the backward one:

\[
y_t = (1 - b) y^B_t + b y^F_t
\]

(7)
A simple example

Because of the expected value $=>$ (1) has an infinite number of solutions (each one corresponding to a particular value of $b$)

Moreover: the expected value in that model under the REH $=$ a weighted average of the past observations (Muth, 1961).

In the example, we have

$$E_t y_{t+1} = (b - 1) \sum_{j=1}^{\infty} \left( \frac{\lambda}{b} \right)^j y_{t+1-j}$$

(8)

Infinite solutions $=>$ infinite way we can set that weights $=>$ $b$ defines the importance the agents give to the past data, both in absolute terms ($b$ vs 1), and in relative terms.
The stability criteria

Is the stability criteria sufficient to identify a unique path?

1. If $\lambda < 1$  
   \textit{NO}  
   indeterminacy

2. If $\lambda > 1$  
   \textit{YES}  
   determinacy, by imposing $b = 1 = \text{f.l. solution}$

However, 3 problems:
- identification: $\lambda$ does not appear in the solution;
- Cochrane (2011,p.6): "\textit{Transversality conditions can rule out real explosions, but not nominal explosions}".
- saddle path has zero measure
Benhabib and Farmer (1999, p.390):

"Sunspot equilibria can often be constructed by randomizing over multiple equilibria of a general equilibrium model, and models with indeterminacy are excellent candidates for the existence of sunspot equilibria since there are many equilibria over which to randomize."

Introducing sunspot equilibria

We have infinite equilibria because:

- there is an infinite number of ways of forming expectations
- all of them coherent with the REH
- parametrized by $b$,

hence natural to introduce sunspots randomizing over $b$:

$$b_t = b_t(\zeta_t) \quad (9)$$

where $\zeta_t$ i.i.d., orthogonal to the fundamental shocks $\omega_s \ (s = 1, 2, ...)$, and $E_t \zeta_t = 0 \ \forall t$. 

Guido Ascari and Paolo Bonomolo

Does Inflation Walk on Unstable Paths? Rational Sunspots and Drifting Parameters

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Example: If $b_t = b_{t-1} + \zeta_t$, then

$$y_t = \alpha_t y_{t-1} - \alpha_t \omega_{t-1} + b_t \omega_t$$

(10)

with $\alpha_t = \lambda \frac{(1 - b_t)}{(1 - b_{t-1})}$.

Sunspot shocks $\Rightarrow$ drifting parameters and stochastic volatility in the rational expectations framework.

Drifting parameters follows naturally if one assumes that agents can modify in every period the expectation formation process $\Rightarrow$ learning
Because of the sources of uncertainty related to the presence of sunspots:

- when a fundamental error hits the economy the initial response of $y_t$ depends on the value of $b_t$;
- even if there is no realization of the fundamental disturbance at $t$, a sunspot shock can change the equilibrium path of $y_t$, changing the coefficient $\alpha_t$. 
At $t - 1$ economic agents expect:

$$E_{t-1}y_t = (1 - E_{t-1}b_t)y_t^B$$ \hspace{1cm} (11)

that is the backward component weighted as in the previous period.

Forecast error:

$$\eta_t = \frac{(E_{t-1}b_t - b_t)}{(1 - E_{t-1}b_t)} E_{t-1}y_t + b_tw_t.$$ \hspace{1cm} (12)

From $t - 1$ to time $t$ two sources of forecast error

- a fundamental error, $\omega_t$;
- a change in the way agents form their expectations, $\zeta_t$. 

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Lubik and Schorfheide (2003, 2004), expectation error:

\[ \eta_t(\omega_t, \zeta_t) = M\omega_t + \zeta^{LS}_t \] (13)

solution:

\[ y_t = \lambda y_{t-1} - \lambda \omega_{t-1} + M\omega_t + \zeta^{LS}_t \] (14)

DIFFERENCE: In our framework, sunspot equilibria are coherent with the REH, in the sense of Muth (1961) => satisfy solution => The "additive" sunspot does not change the way agents form expectations among the infinite possibilities contemplated by rationality.
Rational vs. Irrational Sunspots

LS approach is a case that Muth (1961) describes as a "deviation from rationality" =⇒ can be described by agents over-discounting or under-discounting the most recent observation, as if:

\[ E_t y_{t+1} = f_t u_1 \omega_t + \sum_{i=2}^{\infty} u_i \omega_{t+1-i} \tag{15} \]

where \( f_t \) is a random variable, then the expectation error becomes

\[ \eta_t = b_t \omega_t + (f_t - 1) u_1 \omega_{t-1} \]

\( b_t = M, \) and \( (f_t - 1) u_1 \omega_{t-1} = \zeta_t^{LS} =⇒ \) Lubik and Schorfheide (2003, 2004).
Rational vs. Irrational Sunspots

When sunspot shocks are introduce additively sunspot equilibria result in a bias in the expectations recognized, by Muth (1961) itself, as a "deviation from rationality".

"The way expectations are formed depends specifically on the structure of the relevant system describing the economy" (Muth, 1961, p.316).

In this case, instead, expectations depends on a sunspot that is external to the original model, an hence expectations do not satisfy (8).
"I should like to suggest that expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory. At the risk of confusing this purely descriptive hypothesis with a pronouncement as to what firms ought to do, we call such expectations "rational." " (Muth, 1961, p.316)

Following this quote, we label the mechanism proposed in this paper as "rational sunspots".
Our approach implies a rethinking of the dichotomy between stable and unstable paths. $\alpha_t$ matters, not only $\lambda \Rightarrow b_t$ is not a constant.

Our method provides also a simple **test for stability**: check the *stability criterion* $\Rightarrow$ verify if the parameters in the matrix $b$ corresponding to the unstable eigenvalues, are statistically different from minus one;

Our method provides also a simple **test for determinacy**: a test for determinacy can be based on the likelihood function, that should be a multivariate Normal in the case of determinacy and no sunspots, and not Normal otherwise.
Example: the New Keynesian model

\[ x_t = E_t x_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) + e^x_t \quad \text{(NKIS)} \]
\[ \pi_t = \beta E_t \pi_{t+1} + k x_t + e^\pi_t \quad \text{(NKPC)} \]
\[ i_t = \rho_i i_{t-1} + (1 - \rho_i) \left[ \phi_x x_t + \phi_\pi \pi_t \right] + \epsilon^i_t \quad \epsilon^i_t \sim N(0, \sigma^2_i) \quad \text{(TR)} \]

and

\[ e^x_t = \rho_x e^x_{t-1} + \epsilon^x_t \quad \epsilon^x_t \sim N(0, \sigma^2_x) \]
\[ e^\pi_t = \rho_\pi e^\pi_{t-1} + \epsilon^\pi_t \quad \epsilon^\pi_t \sim N(0, \sigma^2_\pi) \].
We use an econometric strategy to deal with the following issues:

i) the model has stochastic volatility, then the likelihood distribution is not Gaussian;

ii) we are interested in tracking the behavior of $b_t$, that can be considered as a stochastic latent process;

iii) we would like to study the fit of different models, and eventually compare them, during different periods.

Then, the econometric strategy is based on Bayesian methods, in particular on *Particle filtering*, and on *Sequential model monitoring*. 
Model 1

\[ b_t = - \begin{bmatrix} b_{1,t} & 0 \\ 0 & 1 \end{bmatrix} \]

\[ b_{1,t} = b_{1,t-1} + \zeta_t \quad \zeta_t \sim \mathcal{N}(0, \sigma_\zeta^2) \]

The instability due to the eigenvalue outside the unit circle is "killed", as usual in the rational expectation framework \( \Rightarrow \) we take the forward-looking solution \( (b_{2,t} = 1) \) of the difference equation in the Blanchard-Kahn decoupled/diagonalized system associated with the unstable eigenvalue.
## Model 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Distribution</th>
<th>Calibration</th>
<th>1960Q1 - 1979Q3</th>
<th>1979Q4 - 1997Q4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td></td>
<td>0.99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>Unif(0, 1)</td>
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<td>0.6 [0.47 0.71]</td>
<td>0.65 [0.54 0.75]</td>
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<tr>
<td>$\rho_x$</td>
<td>Unif(0, 1)</td>
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<td>0.66 [0.55 0.77]</td>
<td>0.33 [0.15 0.55]</td>
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<tr>
<td>$\rho_\pi$</td>
<td>Unif(0, 1)</td>
<td></td>
<td>0.06 [0.01 0.29]</td>
<td>0.07 [0.01 0.18]</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Ga(0.5,1)</td>
<td></td>
<td>0.17 [0.09 0.3]</td>
<td>0.18 [0.07 0.35]</td>
</tr>
<tr>
<td>$\phi_\delta$</td>
<td>Ga(0.5, 1)</td>
<td></td>
<td>0.031 [0.028 0.039]</td>
<td>0.022 [0.016 0.035]</td>
</tr>
<tr>
<td>$\phi_\pi$</td>
<td>Ga(2, 2)</td>
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<td>1.7 [1.63 1.74]</td>
<td>1.72 [1.62 1.86]</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>InvGa(2, 0.001)</td>
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<td>0.0066 [0.0058 0.0077]</td>
<td>0.0075 [0.0065 0.0091]</td>
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<tr>
<td>$\sigma_x$</td>
<td>InvGa(2, 0.001)</td>
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<td>0.009 [0.007 0.011]</td>
<td>0.01 [0.008 0.013]</td>
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<tr>
<td>$\sigma_\pi$</td>
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<td>0.008 [0.007 0.009]</td>
<td>0.008 [0.007 0.009]</td>
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<tr>
<td>$\sigma_c$</td>
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<td>0.6 [0.5 0.8]</td>
<td>1.24 [1 1.6]</td>
</tr>
</tbody>
</table>

90% credibility interval in brackets
Model 1

The behavior of $b_{1,t}$ over the first sample
the predictive likelihood over the first sample

Predictive likelihood and observed inflation

![Graph showing predictive likelihood and observed inflation over time from 1960 to 1980.](image)
the behavior of \( b_{1,t} \) over the second sample
the predictive likelihood over the second sample
Model 2

\[
b_t = -b_{1,t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
b_{1,t} = b_{1,t-1} + \zeta_t \quad \zeta_t \sim \mathcal{N}(0, \sigma^2_{\zeta})
\]
## Model 2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Distribution</th>
<th>Calibration</th>
<th>Estimates 1960Q1 - 1979Q3</th>
<th>Estimates 1979Q4 - 1997Q4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td></td>
<td>0.99</td>
<td>0.63 [0.45 0.79]</td>
<td>0.65 [0.54 0.76]</td>
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<tr>
<td>(\sigma)</td>
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<td>0.12 [0.03 0.3]</td>
<td>0.54 [0.42 0.65]</td>
</tr>
<tr>
<td>(\rho_i)</td>
<td>Unif(0, 1)</td>
<td></td>
<td>0.16 [0.03 0.22]</td>
<td>0.05 [0.01 0.16]</td>
</tr>
<tr>
<td>(\rho_x)</td>
<td>Unif(0, 1)</td>
<td></td>
<td>0.02 [0.004 0.06]</td>
<td>0.21 [0.05 0.49]</td>
</tr>
<tr>
<td>(\rho_\pi)</td>
<td>Unif(0, 1)</td>
<td></td>
<td>0.32 [0.29 0.37]</td>
<td>0.26 [0.23 0.28]</td>
</tr>
<tr>
<td>(k)</td>
<td>Ga(0.5, 1)</td>
<td></td>
<td>0.32 [0.29 0.37]</td>
<td>0.26 [0.23 0.28]</td>
</tr>
<tr>
<td>(\phi_x)</td>
<td>Ga(0.5, 1)</td>
<td></td>
<td>1.22 [1.15 1.37]</td>
<td>\</td>
</tr>
<tr>
<td>(\phi_\pi)</td>
<td>Ga(2, 2)</td>
<td></td>
<td>1.22 [1.15 1.37]</td>
<td>\</td>
</tr>
<tr>
<td>(\sigma_i)</td>
<td>InvGa(2, 0.001)</td>
<td>0.0057 [0.0049 0.0066]</td>
<td>0.0067 [0.0058 0.0084]</td>
<td></td>
</tr>
<tr>
<td>(\sigma_x)</td>
<td>InvGa(2, 0.001)</td>
<td>0.011 [0.0096 0.013]</td>
<td>0.0096 [0.0083 0.011]</td>
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</tr>
<tr>
<td>(\sigma_\pi)</td>
<td>InvGa(2, 0.001)</td>
<td>0.0073 [0.0064 0.0087]</td>
<td>0.0075 [0.0065 0.009]</td>
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</tr>
<tr>
<td>(\sigma_\zeta)</td>
<td>InvGa(2, 0.01)</td>
<td>0.18 [0.015 0.22]</td>
<td>0.24 [0.2 0.3]</td>
<td></td>
</tr>
</tbody>
</table>

90% credibility interval in brackets
Model 2

The behavior of $b_{1,t}$ over the first sample

$b_{1,t}$ goes far from one exactly when inflation start to surge $\Rightarrow$ Need an unstable model to explain it.
The predictive likelihood in this case does not fall as before.
The odds ratio strongly favours this model compared to the previous one.
Model 2

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the behavior of $b_{1,t}$ over the second sample
Model 2

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