Life Insurance Liabilities: Analysis and Valuation of Embedded Financial Guarantees

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Abstract

In this paper we analyze traditional with-profits life insurance contracts in an option-pricing framework. First we discuss the reserve requirement and the bonus policy, which are typical for such contracts. We show how taking into account these embedded options leads to a Bermudan style path-dependent derivative. We demonstrate how under certain conditions the discretion an insurance company has with respect to the bonus policy can be handled in a contingent-claim-pricing framework.
Abstract

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We demonstrate how under certain conditions the discretion an insurance company has with respect to the bonus policy can be handled in a contingent-claim-pricing framework.

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1 Introduction

From its early start, option-pricing techniques have been applied to the valuation of life insurance contracts (e.g. Brennan and Schwartz (1976)). However, most of these applications only deal with so-called ‘unit-linked’ life insurance contracts; traditional with-profits contracts are largely absent from this literature. The traditional view has been that these contracts do not expose the issuing insurance company to any financial risk, only mortality risk. One reason for this point of view is the fact that an insurance company decides itself on the total return that it credits to the policyholders. This has lead to the perception that an insurance company never committed to any uncertain future payments. A second reason for this viewpoint is that guaranteed returns were usually set at very low, out-of-the-money levels. As a result, even when it was realized that such contracts did indeed include option-like elements, it was argued that the value of these embedded options was negligible. As a result of the significant decline of the interest rates in the 90’s these minimum returns are now often almost at- and in some cases in-the-money, making life insurance companies very well aware of the options embedded in the traditional contracts. See for example Grosen and Jørgensen (2000).

Life-profits life insurance contracts have three important features: the guaranteed minimum return, the reserve requirements and the bonus feature. It is exactly the combination of all these features, together with mortality risk, that characterizes the traditional life insurance contract. The valuation of guaranteed minimum returns has been studied in a unit-linked context and with deterministic interest rates by Brennan and Schwartz (1976), Aase and Persson (1994), Ekern and Persson (1996) and Delbaen (1988). However, none of these contributions deal with the regulatory reserve requirements. Briys and de Varenne (1994,1997), Persson and Aase (1997) and Persson (1998) allow for stochastic interest rates but do not discuss reserve requirements either. In all of the above contributions life insurance contracts are essentially European-style contracts with more or less complex, possibly path-dependent pay-offs. Grosen and Jørgensen (1997) add a multi-period flavor by modelling a contract as an American version of the Brennan and Schwartz (1976) model. Grosen and Jørgensen (2002) model regulatory reserve requirements by means of a barrier option, but do not consider the periodic bonus.

Taking into account the various option features for the valuation of with-profits life insur-
ance contracts is clearly important from a risk-management perspective and embedded value calculations. If the balance sheet of a life insurance company is to give an accurate picture of the value of its liabilities, then a correct valuation of the embedded option elements becomes a necessity.

A completely different reason to correctly take into account the various option features in life insurance contracts is related to the discretion an insurance company has in setting the bonus or excess return that is periodically credited to a contract. If life insurance companies have (cohort of) contracts with different guaranteed returns, then a correct valuation of the contracts is necessary to obtain a fair allocation of excess returns to the different contracts. Especially since, as Grosen and Jorgenson (2002) point out, the naive strategy of allocating a single total return, in excess of the highest guaranteed return, to all contracts might not be a financially sound possibility in periods of low interest rates.

The remainder of the paper is organized as follows. Section 2 gives a discussion of the one-period contract, which allows us to introduce some concepts that will be used throughout the paper. In Section 3 we move on to the multi-period contract, where we analyse how the various option elements derive from the guaranteed return. Section 4 deals with a detailed discussion of the bonus policy, and special attention is given to how the discretion an insurance company has in setting the bonus can be dealt with in a contingent-claim pricing framework. In Section 5 we introduce the Least Squares Monte Carlo Method of Longstaff and Schwartz (2001) which serves as a computational tool later on, and we derive some results under the Black-Scholes-Merton assumptions. Section 6 contains a number of numerical examples illustrating some of the properties of contract such as the of fair-value condition. Finally, Section 7 contains the conclusions. From now on we will assume that the insurance contracts are standard with-profit contracts.

2 The One-period Contract

In this section we introduce the various aspects of an insurance contract such as the premium principle and the concept of fair-value contracts by means of the one-period contract.

In the remainder of the paper we assume the existence of a traded asset \( A(t) \) and a filtration \((\mathcal{F}_t)\), generated by the process \( A(t) \) and the term structure of interest rates.
2.1 The Liabilities

We assume that a life insurance company guarantees a minimum return $r_G$ over the life of the contract. More precisely, for an initial investment $D(t_0)$ the pay-off at maturity $B(T)$ is given by:

$$B(T) = D(t_0) \left[ e^{r_G(T-t_0)} + \gamma \max \left\{ \frac{A(T)}{A(t_0)}, e^{r_G T} \right\} \right],$$

(1)

with $t_0$ the time of inception of the contract, $t_n$ the maturity date and $0 < \gamma \leq 1$. For the time being we assume that $\gamma$ is a constant. We assume, that the underlying portfolio $A(t)$ consists of traded assets. The minimum pay-off $\overline{B}$ guaranteed at $t_n$ by such a contract is given by:

$$\overline{B} = D(t_0)e^{r_G(t_n-t_0)}.$$

This type of contract has been widely studied in the literature, see for instance Brennan and Schwartz (1976), Aase and Persson (1994) and Ekern and Persson (1996) for the single premium case and Delbaen (1990) for a model with periodic premium payments.

2.2 Premium Principle and Fair-value Contracts

The premium $\pi$ for a contract as described above is typically given by an expression of the following type:

$$\pi = e^{-(T-t_0)r_G}\overline{B} = D(t_0)\frac{1}{1-\beta}.$$

(2)

with the risk-loading $\beta$ such that $0 \leq \beta < 1$. That is, the premium is obtained by discounting the minimum pay-off $\overline{B}$ at the guaranteed rate of return and grossing up this amount with a factor $1/(1-\beta)$. Put differently, the premium is obtained by grossing up the reserve value or nominal value $D(t_0)$. For a more detailed discussion of premium principles for life insurance contracts we refer the reader to Gerber (1997).

Because of the above premium-principle a fair-value contract will only be obtained for specific choices for the values of the various parameters. With a fair-value contract we mean a contract for which the premium $\pi$ paid by the policyholder is equal to the value of the contract. For a contract to be a fair-value contract the following condition needs to hold:
with \( P(t_0, T) \) the price at \( t_0 \) of a zero-coupon bond with maturity date \( T \) and where \( ET \) indicates that the expectation is taken under the \( T \)-forward measure.

Observe that there is a direct trade-off between the value for the participation rate \( \gamma \) and the risk-loading \( \beta \). If the total realized return on the assets is credited to the contract, i.e. \( \gamma = 1 \), then the contract can only be a fair-value contract for a value of the risk-loading \( \beta \) greater than zero. More precisely, there is a maximum \( \beta_{\text{max}} \) value for \( \beta \) for which the contract will be a fair-value contract if \( \gamma \) is set to one. For a value for \( \beta \) larger than \( \beta_{\text{max}} \) a fair-value contract can only be obtained if also other features of the contract are changed: e.g. the dynamics of \( A(t) \) or the guaranteed return \( r_G \). Equivalently, given the other features of the contract, there exists a minimum value \( \gamma_{\text{min}} \) for the participation rate for which the contract is a fair-value contract if \( \beta = 0 \). For a choice of the participation rate below \( \gamma_{\text{min}} \) the contract can only be made a fair-value contract if also other features of the contract (i.e. other than \( \beta \)) are changed.

3 The Multi-Period Contract

In this section we discuss the multi-period version of the contract introduced in Section 2. Here, the minimum return \( r_G \) is guaranteed over each of a number of sub-periods, which gives rise to a bonus option and minimum-value requirement/surrender feature. First, we discuss these two option elements separately, before combining them into a single contract.

3.1 The Periodic Bonus

Unlike before, we now assume that the minimum return \( r_G \) does not only have to be guaranteed over the entire maturity of the contract (giving rise to the minimum pay-off \( B \)), but that it has to be guaranteed over each of a number of sub-periods \((t_i, t_{i+1})\) defined by the \( n \) bonus (reserve) dates \( \{t_i\}_{i=1}^{n} \), with \( t_n = T \). As such, at each of these bonus dates the contract is credited the guaranteed return \( r_G \) and an excess return determined by the realized return on the assets \( A(t_i) \). Although at the contract level insurance companies do not guarantee any future excess returns to policyholders. However, in most countries regulations oblige insurance companies to assign or pay out a minimum fraction of the realized return on their
investments on a periodic (yearly) basis.

We assume that a bonus is credited by creating a bonus contract (in a way to be made more precise below), which in turn increases the minimum pay-off \(B\) at maturity. This type of bonus policy is rather common for traditional with-profits life insurance contracts.

Before we turn to the details of such a bonus policy, we point out that two comments can be made with respect to bonus policies. First of all, a bonus policy could be largely cosmetic. An insurance company could sell a contract with a minimum return \(r_G\) well below the prevailing market interest rate for the maturity of the contract and invest in government bonds with a similar maturity. In such a case, bonuses would simply gross up the guaranteed return to this market interest rate. Secondly, Brennan (1993) shows that a non-cosmetic bonus policy that changes the minimum guaranteed amount can not be optimal in a complete and frictionless market. However, it seems reasonable to assume that for individual policy holders transaction costs related to life insurance contracts are non-trivial. Hence, the existence of non-cosmetic bonus policies that change the guaranteed amount \(B\) need not be sub-optimal. We will now discuss in more detail this type of bonus policy.

At a bonus date \(t_{m+1}\) the guaranteed return \(r_G\) is credited to the contract by increasing the previous reserve value \(D(t_m)\):

\[
D(t_{m+1}^-) = D(t_m)e^{r_G(t_{m+1}-t_m)}. \tag{4}
\]

The notation \(D(t_{m+1}^-)\) indicates that we consider the reserve value at \(t_{m+1}\) before bonus allocation. If over the interval \((t_m, t_{m+1})\) the ratio \(\frac{A(t_{m+1})}{A(t_m)}\) is larger then \(e^{(t_{m+1}-t_m)r_G}\) a bonus will be assigned.

More precisely, the bonus amount is equal to:

\[
\begin{align*}
\gamma \left( \frac{A(t_{m+1})}{A(t_m)} - e^{(t_{m+1}-t_m)r_G} \right) D(t_m), & \quad \text{if } \frac{1}{t_{m+1}-t_m} \ln \left[ \frac{A(t_{m+1})}{A(t_m)} \right] > r_G \\
0, & \quad \text{if } \frac{1}{t_{m+1}-t_m} \ln \left[ \frac{A(t_{m+1})}{A(t_m)} \right] \leq r_G. \tag{5}
\end{align*}
\]

As already mentioned, we assume that in case a bonus is assigned an insurance company uses the amount of the bonus to finance a new contract with the same features (maturity date, level of \(r_G\), underlying,...) as the original one. This means that in case a bonus is paid, the bonus amount \(P_b\) given by
\[ P_b = \gamma \left[ \frac{A(t_{m+1})}{A(t_m)} - e^{(t_{m+1}-t_m)rg} \right] + D(t_m), \] (6)

is used as a premium. Reversing the premium principle as given by equation 2 gives \( D_b(t_{m+1}) \), the required reserve (nominal amount) at \( t_{m+1} \) of this new contract:

\[
D_b(t_{m+1}) = (1 - \beta)P_b \\
= (1 - \beta)\gamma \left[ \frac{A(t_{m+1})}{A(t_m)} - e^{(t_{m+1}-t_m)rg} \right] + D(t_m). \] (7)

The guaranteed amount \( B_b(t_{m+1}) \) in turn is equal to:

\[
B_b(t_{m+1}) = D_b(t_{m+1})e^{-rg(t_n-t_{m+1})} \\
= (1 - \beta)\gamma \left[ \frac{A(t_{m+1})}{A(t_m)} - e^{(t_{m+1}-t_m)rg} \right] + D(t_k)e^{-rg(t_n-t_{m+1})}. \] (8)

We see that by allocating a bonus in this way, the reserve value \( D(t_{m+1}) \) and the guaranteed minimum amount \( B(t_{m+1}) \) after bonus allocation at \( t_{m+1} \) are given by the following equations:

\[
D(t_{m+1}) = D(t_m^-) + D_b(t_{m+1}) \\
= \left( e^{rg(t_{m+1}-t_m)} + (1 - \beta)\gamma \left[ \frac{A(t_{m+1})}{A(t_m)} - e^{(t_{m+1}-t_m)rg} \right] \right) + D(t_m).
\]

and:

\[
B(t_{m+1}) = B(t_m) + B_b(t_{m+1}) \\
= D(t_{m+1})e^{(t_n-t_k)rg}.
\]

From this discussion we see that this type of bonus policy turns the life insurance contract into a so-called compounding guaranteed return contract, see for instance Miltersen and Persson (1999). Therefore, the above analysis yields the following proposition.

**Proposition 1** For a contract with maturity \( T \), bonus dates \( \{t_k\}_{k=1}^n \) (\( t_n = T \)) and the type of bonus policy as described above the final pay-off at maturity is that of a compounding
guaranteed return contract. The reserve value or nominal amount after bonus allocation at time $t_k$ is given by:

$$D(t_k) = D(t_0) + \sum_{i=1}^{k} D_b(t_i)$$

and the pay-off at maturity is given by:

$$B(t_n) = D(t_0) \prod_{i=1}^{n} \left( e^{(t_n-t_{i-1})r_G} + (1-\beta)\gamma \left[ \frac{A(t_i)}{A(t_{i-1})} - e^{(t_i-t_{i-1})r_G} \right]_+ \right).$$

We see that because of the specific bonus policy, issuing a so-called bonus contract, the actual participation rate for the contract is now given by:

$$(1-\beta)\gamma.$$

That a fraction lower than the original participation rate $\gamma$ of the realized excess return is credited to the contract can be seen as compensation for the fact that the minimum return $r_G$ is now guaranteed over each of the sub-periods.

Having discussed this interpretation of the guaranteed return $r_G$, we now move on to how the guaranteed return can also be seen as creating a surrender option.

### 3.2 The Minimum-value Requirement and the Surrender Feature

The one-period contract in Section 2 guarantees a minimum pay-off $\overline{B}$ at maturity. This minimum pay-off is closely linked to the guaranteed return $r_G$ as shown by the following equation:

$$\overline{B} = e^{r_G(t_n-t_0)}D(t_0).$$

That is, guaranteeing a minimum pay-off $\overline{B}$ at maturity date $T$ or guaranteeing a minimum return $r_G$ over the period $(t_0, t_n)$ is equivalent. However, in a multi-period setup in which the "guarantee" applies periodically over the life of the contract these two interpretations are no longer equivalent. In section 3.1.1. it was assumed that in the multi-period version the
minimum return $r_G$ is not only guaranteed over the entire maturity of the contract, but over each of a number of sub-periods.

An alternative is to require that a minimum amount is guaranteed at each of a number of reserve dates $\{t_k\}_{k=1}^n$. That is, just as the minimum return $r_G$ guarantees a minimum pay-off at maturity, it also guarantees a minimum pay-off at each of the reserve dates $\{t_k\}_{k=1}^n$. Analogue to equation 11, the guaranteed amount $D(t_k)$ at a reserve date $t_k \leq t_n$ is given by:

$$D(t_k) = D(t_0)e^{r_G(t_n-t_0)}.$$  \hspace{1cm} (12)

Such a minimum-value or reserve requirement is quite common with traditional (with-profits) life insurance contracts. Usually the policyholder also has the possibility to terminate the contract at each of the reserve dates in return for the reserve value given by equation 12. Therefore, if we see the multi-period version of the contract as a contract that periodically guarantees a minimum amount, then the multi-period version of the life insurance contract is a Bermudan style contract with as exercise dates the reserve dates $\{t_k\}_{k=1}^n$ and time-varying strike prices $\{D(t_k)\}_{k=1}^n$, as given by equation 12.

The minimum-value requirement given by equation 12 is the same as the one used by Grosen and Jorgensen (2002). However, whereas in our case it is part of the contract between the policyholder and the insurance company, in Grosen and Jorgensen (2002) equation 12 determines a default or solvency boundary. They start from the same basic contract introduced in Section 2, but allow for the possibility that an insurance company can become insolvent, default on its obligations, over the life-time of the contract, with a minimum-value boundary similar to equation 12 being used as an early warning system. If the value of the assets underlying the contract drop below the boundary, regulatory authorities intervene and the assets of the insurance company are handed over to the policyholders.

### 3.3 Combining the Reserve Requirement with the Periodic Bonus

Here we combine the two option features implied by the guaranteed return $r_G$ into a single contract. That is, we add the minimum-value requirement (surrender option) discussed in Section 3.2 to the contract discussed in Section 3.1. That is, the policyholder can terminate (surrender) the contract at each of the reserve dates $\{t_k\}_{k=1}^{n-1}$ in return for the reserve value $D(t_k)$, i.e. after a bonus has been allocated. This combination of the two option elements
turns the traditional with-profifs life insurance contracts into a Bermudan guaranteed return
contract as analyzed by Simon (2004).

The bonus-mechanism that was introduced in Section 3.1 results in a contract that guar-
antees a minimum return $r_G$ over each sub-period. However, the fact that a minimum return
is guaranteed on the investment does not imply that after bonus allocation at a bonus date
t_k the contract is worth at least its reserve value $D(t_k)$. If (over future periods) not the entire
realized return on the asset $A$ is credited to the contract then the contract can be worth less
then the reserve value $D(t_k)$. Whether or not on a reserve date $t_k$ a contract is worth the
reserve value $D(t_k)$ clearly depends on the values of $r_G$, $\beta$ and $\gamma$. We have the following
result on the optimal exercise decision, similar to a result in Simon (2004).

**Theorem 1** If for a contract as described above the total return on the underlying asset
is credited to the contract, that is the participation rate $\gamma$ is equal to one and there is no
risk-loading: $\beta = 0$, then the minimum-value requirement can never be binding at any of the
surrender dates and surrendering the contract prior to maturity can never be optimal.

**Proof:**

The value of the required reserve at time $t_k$ is given by $D(t_k)$. Investing this amount in
the underlying (traded) asset results in a pay-off at maturity equal to:

$$D(t_k) \frac{A(t_n)}{A(t_k)}.$$  \hspace{1cm} (13)

However, the pay-off of the contract at maturity is given by:

$$D(t_k) \prod_{i=k+1}^{n} \max \left( \frac{A(t_i)}{A(t_{i-1})} e^{(t_i-t_{i-1})r_G} \right).$$  \hspace{1cm} (14)

Since the pay-off given by equation 13 is dominated that of equation 14 the contract is
always worth more than $D(t_k)$. Therefore, the minimum-value requirement will never be binding.

Having finished the description of the multi-period contract, we now move on to the
concept of a fair-value contract in the multi-period framework.
4 The Bonus Policy and Fair-value Contracts

In this section we allow the value of the participation rate $\gamma$ to change over time. This gives an insurance company the possibility to opt for a certain bonus policy. We first discuss the deterministic type of bonus policy before moving on the stochastic version.

4.1 A Deterministic Bonus Policy

The initial fair-value condition given by equation 3 ensures that the premium $\pi$ is equal to the market value $V(t_0)$ of the contract at time $t_0$. Combining this with the premium principle as given by equation 2 leads to:

$$V(t_0) = \frac{D(t_0)}{1 - \beta}. \quad (15)$$

That is, the fair-value condition creates a direct relation between the reserve level and the value of the contract at time $t_0$. Clearly, the fair-value condition will need to be taken into account when assigning values to the various parameters of the contract. Since the loading $\beta$ is positive, equation 15 implies that the value of contract at inception is at least equal to the initial reserve level $D(t_0)$. Note that the minimum-value requirement discussed in Section 2.3.2 creates a similar condition. At each reserve date $t_i < t_n$ one has that:

$$V(t_i) \geq D(t_i). \quad (16)$$

So far, we have assumed that the participation rate $\gamma$ is constant over the life of the contract. However, whereas due to regulatory requirements the guaranteed return $r_G$ and the loading $\beta$ are often fixed for the life-time of the contract, the value of $\gamma$ is at the discretion of the insurance company. Moreover, all previous results are still obtained if, after having been set at $t_0$, $\gamma$ can be reset, either in a deterministic or a stochastic way, at $t_1, ..., t_{n-1}$. If we now allow the value for the participation rate $\gamma$ to be reset at each reserve date $t_i$ the above condition on the value of the contract could be met by resetting the value of $\gamma$ at $t_i$.

Let us assume that an insurance company can reset the participation rate $\gamma$ at each reserve date $t_i < t_n$. In this case, the participation rate could be set at each reserve date $t_i$ (for the period $(t_i, t_{i+1})$) such that the following condition holds at $t_i$:
for a series \((\beta_i)_i \geq 0\) and with \(\beta_0\) equal to the original loading as in equation 15, and with \(\gamma_k(\omega)\) measurable with respect to \(\mathcal{F}_k\). Again, for the time being we assume that the \(\beta_i\)'s are constants. The notation \(\gamma_k(\omega)\) indicates that the participation rate is stochastic and can be reset at each reserve date. Clearly, the above condition implies:

\[
V(t_i) = \frac{D(t_i)}{1 - \beta_i},
\]

That is, as a result of resetting the participation rate a type of fair-value condition is met at each reserve date \(t_i\), with the loading at \(t_i\) equal to \(\beta_i\). Which implies that the surrender option becomes redundant. Observe that the value \(\gamma_i(\omega)\) for the participation rate chosen at \(t_i\) only applies over the period \((t_i, t_{i+1})\) as at \(t_{i+1}\) the participation rate is reset again. This leads to the following result:

**Proposition 2** A deterministic bonus policy determined by a series of loadings \((\beta_i)_i\) as discussed above is equivalent to setting the participation rate \(\gamma_i(\omega)\) at each reserve date \(t_i\) such that:

\[
E^Q \left[ e^{-\int_{t_i}^{t_{i+1}} r(u) \, du} \prod_{k=i+1}^{n-1} \left( e^{(t_k-t_{k-1})r_G} + (1 - \beta) \gamma_k(\omega) \left[ \frac{A(t_k)}{A(t_{k-1})} - e^{(t_k-t_{k-1})r_G} \right]_+ \right) \mid \mathcal{F}_{t_i} \right] = \frac{1}{w_i},
\]

where the \(w_i\)'s are given by:

\[
w_i = \frac{1 - \beta_{i+1}}{1 - \beta_i} \quad \text{if} \quad i < n - 1,
\]

and:

\[
w_{n-1} = 1 - \beta_{n-1}.
\]

**Proof:**

Because of equation 22 the above proposition clearly holds for \(i = n - 1\). The following induction argument completes the proof.
Equation 17 can be rewritten as:

\[
E^Q \left[ e^{\int_{t_i}^{t_{i+1}} r(u) du} R_i \right] E^Q \left[ e^{\int_{t_i}^{t_{i+1}} r(u) du} \prod_{k=i+1}^{n} R_k \ \mathcal{F}_{t_{i+1}} \right] |_{\mathcal{F}_{t_i}} = \frac{1}{1 - \beta_i}, \tag{23}
\]

with:

\[
R_k = e^{(t_k - t_{k-1}) r_G} + (1 - \beta) \gamma_k(w) \left( \frac{A(t_k)}{A(t_{k-1})} - e^{(t_k - t_{k-1}) r_G} \right)_+. \tag{24}
\]

If we assume that equation 19 holds at \(t_{i+1}\), the equation 23 is equivalent to:

\[
E^Q \left[ e^{\int_{t_i}^{t_{i+1}} r(u) du} R_i \frac{1}{1 - \beta_{i+1}} \mathcal{F}_{t_i} \right] = \frac{1}{1 - \beta_i}.
\]

Therefore, equation 19 holds at \(t_i\) for the value of \(\gamma_i(w)\) that is the solution of:

\[
E^Q \left[ e^{\int_{t_i}^{t_{i+1}} r(u) du} \left( e^{(t_{i+1} - t_i) r_G} + (1 - \beta) \gamma_i(w) \left[ \frac{A(t_{i+1})}{A(t_i)} - e^{(t_{i+1} - t_i) r_G} \right]_+ \right) \right] |_{\mathcal{F}_{t_i}} = \frac{1 - \beta_{i+1}}{1 - \beta_i}.
\]

Notice that equations 21 and 22 are equivalent with:

\[
\frac{1}{1 - \beta_i} = \prod_{k=i}^{n-1} \frac{1}{w_k} \geq 1 \text{ for all } 0 \leq i < n. \tag{25}
\]

With this type of bonus policy an insurance company uses the initial loading \(1/(1 - \beta)\) as an additional reserve, which it can use or spend over the various sub-periods of the contract. That is, at \(t_0\) a solvency reserve equal to \(D(t_0)/(1 - \beta)\) is created. Between two reserve dates the value of this reserve changes in lock-step with the assets or replicating portfolio underlying the contract, assuming that it is invested in the same assets. At a reserve date \(t_i < t_n\) an insurance company allocates a fraction of the solvency reserve to the contract. The higher this fraction \(1/w_i\), the higher the participation rate \(\gamma_i(w)\) that applies over the period \((t_i, t_{i+1})\).

Notice that the discretion that the insurance company has in managing the liability side of the contract does not lay in the changing the value of the participation rate \(\gamma\), but in how it has decided to distribute the up-front loading \(1/(1 - \beta)\) over the various sub-periods of the
contract, i.e. on the initial choice of the $w_i$'s. Given the values for the $w_i$'s, the dynamic bonus policy $(\gamma_i(\omega))_i$ is completely determined. However, to assume that the different loadings $w_i$'s would all be set or known at inception might be to strict an assumption. This brings us to the next section on contingent bonus policies.

4.2 Contingent Bonus Policies

In the previous section we assumed that the up-front loading $1/(1 - \beta)$ is distributed in a deterministic way over the different sub-periods of the contract by means of a series of weights $w_i$'s. As a result, the dynamic bonus policy is in a certain sense pre-determined. A more realistic assumption would be to make the weights $(w_i)_i$ stochastic, just as the participation rate $\gamma$. More precisely, here we assume that the weights $(w_i)_i$ are contingent on the evolution of the underlying asset and of the term structure of interest rates. As a result, the participation can be driven by the level of interest rates or past realized returns.

However, how to specify such a contingent policy for the distribution of the reserve loading $1/(1 - \beta)$ is in general not obvious, since one needs that $0 < w_i(\omega)$ for all $i$, and a stochastic version of condition given by equation 25 will need to be met. More precisely one needs that for $i = 0, ..., n - 1$:

$$E^Q \left[ \prod_{k=i}^{n-1} \frac{1}{w_k(\omega)} \bigg| \mathcal{F}_{t_i} \right] = \frac{1}{1 - \beta_i}. \quad (26)$$

An example of what such a contingent bonus policy could look like is given below.

**Example:**

Consider the following choice for $w_i(\omega)$:

$$w_i(\omega) = c_i(\omega)(1 - \beta)^{1/n} \quad (27)$$

with $c_0$ some constant, and the coefficients $c_i$ for $i = 1, ..., n - 1$ are given by:

$$c_i(\omega) = e^{\int_{t_{i-1}}^{t_i} r(u) du} P(t_{i-1}, t_i) c_0^{-1/(n-1)}. \quad (28)$$

The above specification for the $w_i(\omega)$'s meets the condition of equation 26 can be seen as follows. Investing one Euro at $t_i$ in a one-year discount bond and rolling over the investment until $t_{n-1}$ leads to the following pay-off at $t_{n-1}$:
\[
\prod_{k=1}^{n-1} \frac{1}{P(t_{k-1}, t_k)}
\]

Since such an investment is worth the initial outlay of one Eur, that is:
\[
E \left[ \prod_{k=i}^{n-1} \frac{1}{P(t_k, t_{k+1})} \bigg| \mathcal{F}_{t_0} \right] = E \left[ \frac{1}{P(t_{i-1}, t_i)} \prod_{k=i}^{n-1} \frac{1}{P(t_k, t_{k+1})} \bigg| \mathcal{F}_{t_i} \right] = 1.
\]

From which it follows that the \( w_i(\omega) \)'s meet the condition given by equation 26.

The interpretation here is that \( e^{-\int_{t_{i-1}}^{t_i} r(u)du} / P(t_{i-1}, t_i) \) is an adjusted measure for the change in the spot rate over the period \((t_{i-1}, t_i)\). If at \( t_{i-1} \) interest rates are high, this will be reflected already in the value of \( P(t_{i-1}, t_i) \), and the (expectedly) high value \( e^{-\int_{t_{i-1}}^{t_i} r(u)du} \) is scaled down. Therefore, the value of the ratio \( e^{-\int_{t_{i-1}}^{t_i} r(u)du} / P(t_{i-1}, t_i) \) can be seen as an adjusted measure of change in the spot rate over the period \((t_{i-1}, t_i)\). The more interest rates have gone up (down) the lower (higher) the ratio \( e^{-\int_{t_{i-1}}^{t_i} r(u)du} / P(t_{i-1}, t_i) \) will be, and the higher (lower) the fraction \( 1/w_i \) of the solvency reserve that is allocated to the contract at \( t_i \).

In the next section we allow the bonus policy to be defined in a more general way, which gives more room for an insurance company’s discretion setting the bonus.

### 4.3 Discretionary Bonus Policies

So far the fraction \( 1/w_i \) of the solvency margin that is allocated to the contract at a reserve date \( t_i \) is contingent on the underlying assets and the term structure. As such, the insurance company still has very little discretion in setting the bonus policy. An insurance company only gets to make a choice at inception, how the \( 1/w_i \)'s and the participation rate evolve afterwards is determined by the evolution of the underlying assets and interest rates. Here we introduce a type of bonus policy that is no longer contingent on the underlying assets or term structure of interest rates. However, we show that if an adapted version of the condition given by equation 25 is met, pricing a contract with such a ‘non-contingent’ feature creates no problem in a contingent-claim pricing framework.

With a non-contingent or discretionary bonus policy we mean that we no longer assume that for each \( i = 0, ..., n - 1 \) the weight \( w_i \) is contingent on the underlying assets and term structure. Dropping this assumption means that the \( w_i \)'s are now an additional source of uncertainty. We also introduce a second filtration \( \mathcal{G}_t \) for which we have that: \( w_i \) is \( \mathcal{G}_{t_i} \)-measurable and that \( \mathcal{F}_{t_i} \subseteq \mathcal{G}_{t_i} \) for each \( i = 0, ..., n \). That is, an insurance company’s decision
for \( w_i \) is based on the evolution of the underlying asset and other information, such as economic variables but also its market position.

Again, we need a stochastic version of the condition given by equation 25. Here we make the following assumptions. For all for all \( i = 1, ..., n-1 \):

\[
\forall \omega : \prod_{k=i}^{n-1} \frac{1}{w_k} \geq 1,
\]  
(29)

and:

\[
\forall \omega : \prod_{k=0}^{n-1} \frac{1}{w_k} = \frac{1}{1 - \beta},
\]  
(30)

Equation 29 guarantees that the minimum-value requirement is met at each reserve date \( t_i > t_0 \), and equation 30 means that over the life-time of the contract the entire solvency marging \( 1/(1 - \beta) \) is allocated to or invested in the insurance policy. Observe that because of equation 30 the product \( \prod_{k=i}^{n-1} 1/w_k \) is known at \( t_i \), since:

\[
\prod_{k=i}^{n-1} \frac{1}{w_k} \times \prod_{k=0}^{i-1} \frac{1}{w_k} = \frac{1}{1 - \beta},
\]

and both the second factor on the right-hand side and the level of the loading \( \beta \) are known at \( t_i \). We now turn to the valuation of a contract with such a discretionary bonus policy; for which we have the following result:

**Proposition 3** For a contract with a discretionary bonus policy \( (w_i)_i \) that meets the conditions given by equations 29 and 30 the value \( V(t_i) \) after bonus allocation at a reserve date \( t_i \) is given by:

\[
V(t_i) = \frac{D(t_i)}{1 - \beta_i},
\]  
(31)

with:

\[
\frac{1}{1 - \beta_i} = \prod_{k=i}^{n-1} \frac{1}{w_k}.
\]  
(32)

Before we give the proof of this proposition we observe that the essential thing about it is the fact that equation 31 is the same as in the case of a contingent bonus policy. As such, the additional uncertainty introduced into the set-up by the discretionary bonus policy does not seem to be priced. The proof below is essentially an argument for why this is the case.
Proof:

We proof equation 30 by induction.

Step 1: \( i = n - 1 \)

Observe that at \( t_{n-1} \) the participation rate \( \gamma_{n-1}(\omega) \) takes on the value such that:

\[
E^Q \left[ e^{-\int_{t_{n-1}}^{t_n} r(u) du} \left( e^{(t_n-t_{n-1})r_G} + (1-\beta)\gamma_{n-1}(\omega) \left[ \frac{A(t_n)}{A(t_{n-1})} - e^{(t_n-t_{n-1})r_G} \right]_+ \right) \right] \bigg| F_{t_{n-1}} \]  
= \frac{1}{w_{n-1}}.

Since we have that \( w_{n-1} = 1 - \beta_{n-1} \), it follows that the value of the contract at \( t_{n-1} \) is given by:

\[ V(t_{n-1}) = D(t_{n-1}) \frac{1}{1-\beta_{n-1}}. \]

Step 2: \( i + 1 \Rightarrow i \)

The value \( V(t_i) \) of the contract at \( t_i \) is given by:

\[ V(t_i) = D(t_i) E^\overline{Q} \left[ e^{-\int_{t_{i+1}}^{t_i} r(u) du} R_i \frac{1}{1-\beta_{i+1}} \bigg| G_{t_i} \right], \]

with:

\[ R_i = \left( e^{(t_{i+1}-t_i)r_G} + (1-\beta)\gamma_i(\omega) \left[ \frac{A(t_{i+1})}{A(t_i)} - e^{(t_{i+1}-t_i)r_G} \right]_+ \right), \]

and \( \overline{Q} \) the appropriate equivalent martingale measure. As at time \( t_i \) a decision is made about the value of \( w_i \), and therefore also about the value of \( \gamma_i(\omega) \). Moreover, as a result of setting the value of \( w_i \) also the value of \( \beta_{i+1} \) is known at \( t_i \). Therefore, there is no “uncertainty” about the factor \( 1/(1-\beta_{i+1}) \) and the measure \( \overline{Q} \) collapses into \( Q \). That is, the value of the contract is given by:

\[
V(t_i) = \frac{D(t_i)}{1-\beta_{i+1}} E^Q \left[ e^{-\int_{t_{i+1}}^{t_i} r(u) du} R_i \bigg| F_{t_i} \right] 
= \frac{D(t_i) \frac{1}{w_i}}{1-\beta_i} 
= \frac{D(t_i)}{1-\beta_i}.
\]
From the proof we see that the additional uncertainty introduced by the non-contingent bonus policy is not priced because the \( w_i \)'s show a specific form of predictibility. Although the values of \( w_{i+1}, \ldots, w_{n-1} \) are not known at \( t_i \), the product \( \prod_{k=i+1}^{n-1} w_k \) is known at \( t_i \). It is this predictability that leads to equation 31.

5 Valuation

In this section we deal with how to value the contract described in Section 3.3, i.e. with both the compounding bonus policy and the surrender feature.

5.1 The Dynamics

We assume that a stock portfolio \( S(t) \) is given of which the physical dynamics are given by:

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma_A dB_1(t),
\]

where \( B_1(t) \) is a standard Brownian motion. Notice that it is assumed that any dividends are re-invested in the portfolio. Further, we assume that the dynamics of the term structure of interest rates are those of the Vasicek (1977) model:

\[
dr(t) = a(b - r(t))dt + \sigma_r dW_2(t),
\]

with \( a, b, \sigma > 0 \) and \( dB_1(t)dB_2(t) = \rho \).

\[
P(t, T) = e^{C(T-t)-B(T-t)r(t)},
\]

\[
B(\tau) = \frac{1 - e^{-a\tau}}{a\tau},
\]

\[
C(\tau) = \frac{(B(\tau) - 1)(a^2b - \sigma^2/2)}{a^2} - \frac{\sigma^2\tau B(\tau)^2}{4a}.
\]

Last, we assume that a fraction \( p \) of the underlying assets \( A(t) \) is invested in the stock portfolio \( S(t) \) and that the remainder is invested in zero-coupon bonds with remaining maturity \( T \). Therefore:
\[ dV(t) = pdA(t) + (1 - p)dP(t, t + T). \]

In this case, the filtration \((\mathcal{F}_t)_{t}\) is the filtration generated by the two Brownian motions \(B_1(t)\) and \(B_2(t)\).

Since in its most general form, without the fair-value condition introduced in Section 4, the multi-period contract is a Bermudan path-dependent derivative, we give a short introduction to the Least Squares method developed by Longstaff and Schwartz (2001) for the valuation of such products.

**5.2 The Longstaff and Schwartz Least-Squares Method**

When using a Monte Carlo method to value an American or Bermudan style derivative, the problem that arises is that of obtaining the continuation value or the value alive for the option at each exercise date. A straightforward approach would be to estimate this value by means of a separate Monte Carlo simulation. However, since this would need to be done for every exercise date on every path at which the option is in the money, such an approach is unfeasible. Longstaff and Schwartz (2001) recognize that an estimate for the continuation value at a given exercise date on a given path can be obtained by regressing the discounted cashflows from continuation on a number of basis functions of the levels of the factors driving the asset dynamics. The continuation value at a given exercise date is given by the fitted value of the regression. The following example illustrates this approach. A similar example is given by Longstaff and Schwartz (2001).

Consider a Bermudan put option with strike price equal to 120, with exercise dates \(t_1 = 1, t_2 = 2\) and \(t_3 = 3\). Let us assume that there is a constant spot rate of 5% that the volatility of the underlying asset is 20% and that \(A(t_0) = 100\). Simulating 8 paths generates the following matrix.
Stock price paths

<table>
<thead>
<tr>
<th>path</th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>98.36</td>
<td>114.50</td>
<td>132.76</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>76.87</td>
<td>111.06</td>
<td>108.53</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>109.97</td>
<td>103.32</td>
<td>105.22</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>113.60</td>
<td>124.01</td>
<td>90.80</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>85.27</td>
<td>95.35</td>
<td>113.76</td>
</tr>
<tr>
<td>6</td>
<td>100</td>
<td>136.10</td>
<td>92.97</td>
<td>82.10</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>136.05</td>
<td>104.36</td>
<td>123.72</td>
</tr>
<tr>
<td>8</td>
<td>100</td>
<td>106.45</td>
<td>109.72</td>
<td>148.40</td>
</tr>
</tbody>
</table>

Along the 8 paths given here the option is in the money at maturity on paths 2, 3, 4, 5, 6 and 7. This leads to the following matrix of cashflows at time $t = 2$.

<table>
<thead>
<tr>
<th>Cashflows at time 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>path</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

The next step is to obtain estimates for the continuation values at $t = 2$ for those paths at which the option is in the money at this exercise date. These are all paths except path 4. The next matrix gives the value of the underlying for the in-the-money paths at $t = 2$ as well as the discounted values of the future cashflows. For instance, along path 3 the pay-off at maturity is 14.78, discounting this value over 1 period at 5% yields 14.06. From the first table one sees that the value of the underlying asset along this path at $t = 2$ is 103.32.

<table>
<thead>
<tr>
<th>Discounted cashflows at time 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>path</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

To obtain the continuation value for each path we first regress the seven $Y$ values on a constant, $X$ and $X^2$. That is we fit the following model: $Y_i = a + b_1 X_i + b_2 X_i^2 + e_i$. This yields
the following estimates for the coefficients: $\hat{a} = 1000.9911$, $\hat{b}_1 = -18.1842$ and $\hat{b}_2 = 0.0828$.

The continuation value for each path is then given by the expected/fitted value $E[Y|X]$. For instance, the continuation value for path 3 is equal to: 1000.9911 – 18.1842 × 103.32 + 0.0828 × 103.32² = 6.10.

<table>
<thead>
<tr>
<th>Exercise decision at time 2</th>
<th>Exercise value</th>
<th>Continuation value</th>
</tr>
</thead>
<tbody>
<tr>
<td>path</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5.59</td>
<td>4.44</td>
</tr>
<tr>
<td>2</td>
<td>9.94</td>
<td>2.74</td>
</tr>
<tr>
<td>3</td>
<td>16.68</td>
<td>6.10</td>
</tr>
<tr>
<td>4</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>24.65</td>
<td>19.93</td>
</tr>
<tr>
<td>6</td>
<td>27.03</td>
<td>26.09</td>
</tr>
<tr>
<td>7</td>
<td>15.64</td>
<td>5.06</td>
</tr>
<tr>
<td>8</td>
<td>10.28</td>
<td>2.61</td>
</tr>
</tbody>
</table>

Comparing the exercise value with the continuation value in the above table shows that exercise at $t = 2$ is optimal along each path under consideration. This results in the following table of cashflows.

<table>
<thead>
<tr>
<th>Cashflows and exercise dates at time 2</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>path</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>–</td>
<td>5.59</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>9.94</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>–</td>
<td>16.68</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>–</td>
<td>0</td>
<td>29.20</td>
</tr>
<tr>
<td>5</td>
<td>–</td>
<td>24.65</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>–</td>
<td>27.03</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>–</td>
<td>15.64</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>–</td>
<td>10.28</td>
<td>0</td>
</tr>
</tbody>
</table>

In order to determine for which of the paths exercise at $t = 1$ is optimal we again compute the present value of the future cashflows for each in-the-money path. The next table gives these values together with the value of the underlying asset.

<table>
<thead>
<tr>
<th>Discounted cashflows at time 1</th>
<th>$X$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>path</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>98.36</td>
<td>5.59 * 0.9512 = 5.23</td>
</tr>
<tr>
<td>2</td>
<td>76.87</td>
<td>9.94 * 0.9512 = 8.50</td>
</tr>
<tr>
<td>3</td>
<td>109.97</td>
<td>16.68 * 0.9512 = 15.87</td>
</tr>
<tr>
<td>4</td>
<td>113.60</td>
<td>29.20 * 0.9512² = 26.42</td>
</tr>
<tr>
<td>5</td>
<td>85.27</td>
<td>24.65 * 0.9512 = 23.45</td>
</tr>
<tr>
<td>6</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>106.45</td>
<td>10.28 * 0.9512 = 9.78</td>
</tr>
</tbody>
</table>
As at $t = 1$ we fit the following model: $Y_i = a + b_1 X_{i1} + b_2 X_{i2} + \epsilon_i$. This yields the following estimates for the coefficients: $\hat{a} = 205.9323, \hat{b}_1 = -4.2623$ and $\hat{b}_2 = 0.0232$. The continuation value for each path is then given by the expected/fitted value $E[Y|X]$. For instance, the continuation value for path 3 is equal to: $205.9323 - 4.2623 \times 109.9732 + 0.02 \times 109.97^2 = 17.39$.

<table>
<thead>
<tr>
<th>path</th>
<th>Exercise</th>
<th>Value alive</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.64</td>
<td>10.73</td>
</tr>
<tr>
<td>2</td>
<td>43.13</td>
<td>14.92</td>
</tr>
<tr>
<td>3</td>
<td>10.03</td>
<td>17.39</td>
</tr>
<tr>
<td>4</td>
<td>6.40</td>
<td>20.76</td>
</tr>
<tr>
<td>5</td>
<td>34.73</td>
<td>10.73</td>
</tr>
<tr>
<td>6</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>13.55</td>
<td>14.71</td>
</tr>
</tbody>
</table>

Comparing the exercise value with the continuation value shows that exercise at $t = 1$ is optimal along paths 1, 2 and 5. This leads to the following matrix of cashflows.

<table>
<thead>
<tr>
<th>path</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.64</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>43.13</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>16.68</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>29.20</td>
</tr>
<tr>
<td>5</td>
<td>34.73</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>27.03</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>15.64</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>10.28</td>
<td>0</td>
</tr>
</tbody>
</table>

Discounting each cashflow back to $t = 0$ and averaging over the 8 paths yields 22.85 as the estimate for the value of the put option.

### 5.3 Some Results under the Black-Scholes-Merton Assumptions

In this section closed-form expressions are obtained under the assumptions underlying the Black-Scholes-Merton formula. That is, we assume that the spot rate $r(t)$ is a constant and that the physical dynamics of $S(t)$ are still given by:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t), \quad \text{(37)}$$

with the assumption that dividends are re-invested. Since there is no interest rate risk in this set-up, we assume that the underlying asset $A(t)$ is completely invested in the stock portfolio.
As such we have $A(t) \equiv S(t)$. For this reason, all formulas in this section use $S(t)$ rather than $A(t)$. Secondly, for reasons that will become clear, we only consider the case that the values for $\gamma$ and $\beta$ are constant.

5.3.1 The Periodic Bonus

Here we consider the contract as described in Section 3.1, with the periodic bonus but no surrender feature. Starting from an initial reserve level $D(t_0)$ the contract is periodically credited the following return:

$$e^{r_G(t_{i+1} - t_i)} + (1 - \beta)\gamma \left[ \frac{A(t_{i+1})}{A(t_i)} - e^{r_G(t_{i+1} - t_i)} \right].$$

A straightforward application of the Black-Scholes-Merton formula leads to the following result:

**Proposition 4** For a contract as described in this section with $n$ reserve dates $\{t_i\}_{i=1}^n$ one has that for any two reserve dates $t_{n-i}$ and $t_{n-i+1}$ and any $t^*$, $t_{n-i} \leq t^* < t_{n-i+1}$, the value of the contract at $t^*$ is given by:

$$D(t_{n-i}) \left[ g^* + (1 - \beta)\gamma \left( \frac{S(t^*)}{S(t_{n-i})} N(d^*) - g^* N \left( d^* - \sigma \sqrt{t_{n-i+1} - t^*} \right) \right) \right] \prod_{j=0}^{i-2} c_{n-j}, \quad (38)$$

with $c_{n-j}$ given by:

$$c_{n-j} = (1 - \beta)\gamma \left( N(d_{n-j}) - e^{-(r-r_G)(t_{n-j+1} - t_{n-j})} N \left( d_{n-j} - \sigma \sqrt{t_{n-j+1} - t_{n-j}} \right) \right)$$

$$+ e^{-(r-r_G)(t_{n-j+1} - t_{n-j})}.$$ \quad (39)

with:

$$d_{n-j} = \frac{\ln \left[ e^{-r_G(t_{n-j+1} - t_{n-j})} \right] + (r + \sigma^2/2)(t_{n-j+1} - t_{n-j})}{\sigma \sqrt{t_{n-j+1} - t_{n-j}}}.$$ \quad (42)

For the lead factor one has that $g^*$ is equal to:

$$g^* = e^{-r(t_{n-i+1} - t^*)} e^{r_G(t_{n-i+1} - t_{n-i})}$$ \quad (40)

and $d^*$ is given by:

$$d^* = \frac{\ln \left[ S(t^*)/S(t_{n-i}) \right] - r_G(t_{n-i+1} - t_{n-i}) + (r + \sigma^2/2)(t_{i+1} - t^*)}{\sigma \sqrt{t_{n-i+1} - t^*}}, \quad (41)$$
Observe in case $t^*$ is equal to a reserve date, the value of the contract, as given by equation 38, is not a function of the value of the underlying asset $S$. Therefore, at each of the reserve dates $t_i$ the delta of the contract is equal to zero. The delta of the contract in between reserve dates is given by the next proposition, the proof of which is again a straightforward application of the Black-Scholes-Merton results.

**Proposition 5** For a contract as described in this section with $n$ reserve dates $\{t_i\}_{i=1}^n$ one has that for any two reserve dates $t_{n-i}$ and $t_{n-i+1}$ and any $t^*$, $t_{n-i} < t^* < t_{n-i+1}$, the delta of the contract at $t^*$ is given by:

$$
(1 - \beta)^{\gamma} \frac{D(t_{n-i})}{S(t_{n-i})} N(d^*) \prod_{j=0}^{i-2} c_{n-j},
$$

with $d^*$ and $c_{n-j}$ as in the previous proposition.

We see that the value of the contract at each of the reserve dates $t_0, ..., t_{n-1}$ is a constant, even for a constant participation rate $\gamma$. Hence, allowing for a stochastic participation rate wouldn’t contribute anything here.

### 5.3.2 The Minimum-value Requirement

Here we consider the contract as discussed in Section 3.2, under the same Black-Scholes-Merton assumptions. A first result is the next proposition.

**Proposition 6** If $r_G \geq r$ then the minimum-value requirement is redundant. If at $r_G < r$, then at any reserve date $t_i < t_n$ the insurance contract is equivalent to a combination of a compound call option $\gamma D(t_0)/S(t_0) C^{(n-i)} (t_i, S(t_i), \{t_k, \Delta_k\}_{k=i+1})$ and investing the amount $e^{-(t_{k+1} - t_i)} r D(t_{i+1})$ in the money market account.

**Proof:**

A proof of the above proposition can be found in Appendix A.

From the above proposition we see that the minimum-value requirement only matters if the guaranteed return $r_G$ is lower than the riskless rate $r$. This could have been expected, since $r_G \geq r$ would imply that, even when $\gamma = 0$, at each contract date $t_i$ the contract would be worth more then $D(t_i)$. 

23
Before we move on to the valuation of this contract, we first define two classes of matrices.

**Definition 1** For a given set of \( k + 1 \) dates \( t_0, ..., t_k \), the matrix \( \Sigma^{(k,k)} \) is a \( k \times k \) matrix with elements:

\[
\Sigma^{(k,k)}_{ij} = \begin{cases} 
\sqrt{t_i - t_0} & \text{for } i < j \\
1 & \text{for } i = j \\
\sqrt{t_j - t_0} & \text{for } i > j.
\end{cases}
\]  

(43)

**Definition 2** For a given set of \( k + 1 \) dates \( t_0, ..., t_k \), and for an integer \( l \) with \( 2 \leq l < k \) the matrix \( \Sigma^{(k,l)} \) is the \( l \times l \) matrix given by the first \( l \) rows and columns of the matrix \( \Sigma^{(k,k)} \) as given by the previous definition.

With these definitions in mind we can now state the following two theorems.

**Proposition 7** Under the above assumptions, the value at \( t^* \), with \( t_i \leq t^* < t_{i+1} \), of a contract with a Bermudan style reserve requirement as discussed above with initial nominal amount \( D(t_0) \) is given by:

\[
V^{(n-i)}(t^*,S(t^*),\{t_j\}_{j=i+1}^n) = D(t_0) \left[ \frac{S(t^*)}{S(t_0)} \right] \left[ N_{n-i} \left( d^{(n-i)}_1 + \sigma \sqrt{t_{i+1} - t^*}, ..., d^{(n-i)}_{n-1} + \sigma \sqrt{t_n - t^*}; \Sigma^{(n-i,n-i)} \right) - \sum_{j=i+1}^n \Delta_j e^{r(t_j-t^*)} N_{n-j} \left( d^{(n-i)}_1, ..., d^{(n-i)}_{n-j}; \Sigma^{(n-i,n-j)} \right) + e^{r(t_{i+1}-t^*)(r-\sigma^2/2)} \right],
\]  

(44)

where:

\[
d^{(i)}_{n-j} = \frac{\ln \left( \frac{\alpha S(t^*)}{S(t_0) \chi_i} \right) + (r - \sigma^2/2) (t_j - t^*)}{\sigma \sqrt{t_j - t^*}} \quad j = i + 1, ..., n - 1
\]  

(45)

and:

\[
\Delta_i = e^{r\gamma(t_i-t_0)} - e^{r\gamma(t_{i+1}-t_0)}e^{-\gamma(t_{i+1}-t_i)} \quad i = 1, ..., n - 1
\]

\[
\Delta_n = e^{r\gamma(t_n-t_0)}.
\]  

(46)

For all \( i \in \{1...n-1\} \) the value for \( \chi_i \) in equation is given by the solution of the following equation:
\[
V^{(n-i)}(t_i, \chi_i, \{t_j\}_{j=i+1}^n) = e^{(t_i-t_0)} + e^{(t_{i+1}-t_0)}r^G-(t_{i+1}-t_i)r 
\]

Here \(N_{n-i}(x_1, \ldots, x_{n-i}; \Sigma^{(n,n-i)})\) is the multivariate normal probability determined by the \((n-i)\) tuple \((x_1, \ldots, x_{n-i})\) for a multivariate normal distribution with a vector of means equal to the null vector and a covariance matrix \(\Sigma^{(n,n-i)}\). For \(i = 0, \ldots, n-1\), the matrices \(\Sigma^{(n,n-i)}\) are given by definitions 1 and 2 given the \(n+1\) dates \(t_0, t_1, \ldots, t_n\).

Proof:

A proof of the above proposition can be found in Appendix B.

In this case we see that the value of the contract is only independent of the value of the underlying asset \(S\) at inception. That is, only at \(t_0\) is the delta identically equal to zero. The next proposition gives the delta for the contract for any \(t^*\) after inception. The delta of the contract is given by the following proposition.

**Proposition 8** Under the same assumptions as in Proposition 30 and with the same notation, one has that the delta of the contract at \(t^*\), with \(t_i < t^* \leq t_{i+1}\), is given by:

\[
\frac{\partial}{\partial S(t)} V^{(n-i)}(t^*, S(t^*), \{t_j\}_{j=i+1}^n) = \alpha D(t_0) N_n \left( d_1^{(n-i)} + \sigma \sqrt{t_{i+1} - t^*}, \ldots, d_{n-i}^{(n-i)} + \sigma \sqrt{t_n - t^*}; \Sigma^{(n-i,n-i)} \right), \tag{48}
\]

with the \(d_k^{(n-i)}\)'s and the matrices \(\Sigma^{(n-i,n-i)}\) as in the previous proposition.

Proof:

A proof of the above proposition can be found in Appendix B.

From the above proposition one sees that in order to follow a delta-hedging strategy one would need to evaluate a multi-variate normal probability every time the replicating portfolio is rebalanced. At every rebalancing date one also needs to obtain the different arguments for the normal probability. However, the values of the endogenous exercise-boundary variables \((\chi_i)_{i=1}^n\) are invariant, they only get solved for once, at inception of the contract.
5.3.3 Combining Both Features

From Essay 1 we know that under the Black-Scholes Merton assumptions whether or not the minimum-value requirement ends up being in the money at a reserve date \( t_i \), is already known at inception, as illustrated by the next result taken from Essay 1.

**Proposition 9** Under the BSM assumptions one has for a contract as described in Section 2.3.3 one has that the reserve dates \( t_i \) at which the minimum-value requirement is in the money can be determined by the following procedure.

Going back in time starting at the maturity date \( t_n \), the 'first' reserve date at which the reserve requirement is binding, i.e. at which the exercising the embedded put option is optimal, is the reserve date \( t_{n-k} \) determined by the smallest non-zero integer \( i \) for which:

\[
\prod_{j=1}^{i} c_{n-j} > 1,
\]

where for \( j = 1, ..., i \) the coefficient \( c_{n-j} \) is given by:

\[
c_{n-j} = (1 - \beta) \left( \alpha N(d_{n-j}) - e^{-(r-r_G)(t_{n-j+1}-t_{n-j})} N \left( d_{n-j} - \sigma \sqrt{t_{n-j+1}-t_{n-j}} \right) \right) + e^{-(r-r_G)(t_{n-j+1}-t_{n-j})}
\]

with:

\[
d_{n-j} = \ln \left[ \alpha e^{-r_G(t_{n-j+1}-t_{n-j})} \right] + \frac{(r + \sigma^2/2)(t_{n-j+1}-t_{n-j})}{\sigma \sqrt{t_{n-j+1}-t_{n-j}}}. 
\]

Given a reserve date \( t_{n-k} \) at which the reserve requirement is binding. The 'next' reserve date \( t_{n-i} \) (\( i > k \)) at which the reserve requirement is binding is the reserve date \( t_{n-i} \) determined by the smallest integer \( i > k \) for which:

\[
\prod_{j=k}^{i} c_{n-j} > 1,
\]

where for \( j = k + 1, ..., i \) the coefficient \( c_i \) is given by equation 50.

Using the algorithm from the above proposition we can determine the first reserve date \( t_i \) at which the minimum-value requirement is binding. Therefore, the contract collapses into one without the minimum-reserve requirement and with a maturity date equal to \( t_i \).
6 Implementation and Analysis

The Monte Carlo simulations for all tables and figures in this section are based on 8000 (4000+4000 antithetic) paths with 120 time steps per year. For each table or figure all estimates have been generated using a single set of 8000 paths.

6.1 Valuation of Liabilities

Table 2.1 shows the impact of changes the values of the guaranteed return \( r_G \) and the participation rate \( \gamma \) on the value of the liabilities generated by a contract.

The non-variable parameters have the following values: \( r(t_0) = 0.06 \), \( a = 0.2 \), \( b = 0.1 \), \( \sigma_r = 0.03 \), \( \sigma_S = 0.2 \) and \( \rho = -0.2 \). The maturity of the contract is 10 years and \( f = 0.1 \). The remaining fraction of the assets is invested in 8-year discount bonds, a position that is rolled-over at each reserve date.

As expected, the value of the liabilities is increasing in both the guaranteed return \( r_G \) and the participation rate \( \gamma \). For \( \gamma = 1 \) and \( r_G = 0 \) the value of the contract is equal to 1.05 the initial reserve \( D(t_0) \). If we want the value of the contract to be unchanged when we increase the guaranteed return to 0.02 the participation rate will need to be lowered to 0.85. We also see that for \( r_G = 0 \) and \( \gamma = 0.80 \) the liabilities are valued at par, i.e. for such a contract the value is equal to \( D(t_0) \). Note that this is unrelated to whether or not the contract is a fair-value contract, which depends on the value of the loading \( \beta \).

Table 1: Impact of \( r_G \) and \( \gamma \)

<table>
<thead>
<tr>
<th>( r_G )</th>
<th>1.00</th>
<th>0.95</th>
<th>0.90</th>
<th>0.85</th>
<th>0.80</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.05</td>
<td>1.03</td>
<td>1.02</td>
<td>1.01</td>
<td>1.00</td>
</tr>
<tr>
<td>0.01</td>
<td>1.08</td>
<td>1.05</td>
<td>1.04</td>
<td>1.02</td>
<td>1.01</td>
</tr>
<tr>
<td>0.02</td>
<td>1.11</td>
<td>1.09</td>
<td>1.07</td>
<td>1.05</td>
<td>1.04</td>
</tr>
<tr>
<td>0.03</td>
<td>1.15</td>
<td>1.12</td>
<td>1.10</td>
<td>1.08</td>
<td>1.07</td>
</tr>
<tr>
<td>0.04</td>
<td>1.20</td>
<td>1.17</td>
<td>1.14</td>
<td>1.12</td>
<td>1.11</td>
</tr>
<tr>
<td>0.05</td>
<td>1.25</td>
<td>1.22</td>
<td>1.20</td>
<td>1.17</td>
<td>1.15</td>
</tr>
</tbody>
</table>

A second issue that we look into is whether the effect of the initial value of the spot rate \( r_t \) disappears with increasing values of \( T \). Because the spot rate is stationary, one would expect that the impact of the starting value of the spot rate gradually disappears for increasing
values of $T$, the number of years to maturity. Figure 2.1 gives the natural logarithm of the value of the contract (with surrender option) for different values of $T$ and the initial spot rate $r_t$. Here, the non-variable parameters for the dynamics have the following values: $a = 0.2, b = 0.1, \sigma_r = 0.03, \sigma_S = 0.2, \rho = -0.2$. For the contract the fixed parameters are: $\beta = 0.12, \gamma = 0.10$ and $r_G = 0.04$. Again the fraction $f$ is equal to 0.1

Indeed, one observes that for maturities of 20 years and more the log-value of the contract is approximately linear in $T$. Moreover, the ‘asymptotic slope’ of the line also seems to be constant across the various values of $r_t$. However, the lines do not converge, as the value for the spot rate at inception has an initial effect that is not offset anymore later on. This result suggests that one could approximate the far end of long-term contracts as being independent of the term structure.

6.2 The Bonus Policy

Here we illustrate some of the issues about bonus policies. As discussed earlier on, the most basic approach for a bonus policy is to fix the value of the participation $\gamma$ at inception. Table 2.2 gives the value for the loading $\beta$ that leads to a fair-value contract at inception for various values of the participation rate $\gamma$ and for the maturity $T$. The fixed parameters have the following values $r(t_0) = 0.06, a = 0.2, b = 0.1, \sigma_r = 0.03, \sigma_S = 0.2$ and $\rho = -0.2$. The fraction
$f = \text{is equal to } 0.10$. And again the fixed income component of the underlying assets is a rolling-over 8-year discount bond. The guaranteed return $r_G$ is equal to 0.04.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$T$</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td></td>
<td>0.10</td>
<td>0.15</td>
<td>0.19</td>
</tr>
<tr>
<td>0.98</td>
<td></td>
<td>0.10</td>
<td>0.15</td>
<td>0.18</td>
</tr>
<tr>
<td>0.95</td>
<td></td>
<td>0.09</td>
<td>0.14</td>
<td>0.18</td>
</tr>
<tr>
<td>0.93</td>
<td></td>
<td>0.09</td>
<td>0.14</td>
<td>0.17</td>
</tr>
<tr>
<td>0.90</td>
<td></td>
<td>0.08</td>
<td>0.13</td>
<td>0.16</td>
</tr>
<tr>
<td>0.88</td>
<td></td>
<td>0.08</td>
<td>0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>0.85</td>
<td></td>
<td>0.07</td>
<td>0.12</td>
<td>0.14</td>
</tr>
<tr>
<td>0.83</td>
<td></td>
<td>0.07</td>
<td>0.11</td>
<td>0.14</td>
</tr>
<tr>
<td>0.80</td>
<td></td>
<td>0.07</td>
<td>0.11</td>
<td>0.13</td>
</tr>
<tr>
<td>0.78</td>
<td></td>
<td>0.06</td>
<td>0.10</td>
<td>0.12</td>
</tr>
<tr>
<td>0.75</td>
<td></td>
<td>0.06</td>
<td>0.09</td>
<td>0.12</td>
</tr>
<tr>
<td>0.73</td>
<td></td>
<td>0.05</td>
<td>0.09</td>
<td>0.11</td>
</tr>
<tr>
<td>0.70</td>
<td></td>
<td>0.05</td>
<td>0.08</td>
<td>0.10</td>
</tr>
</tbody>
</table>

From Table 2.2 we see that for a fixed value of $T$ we have the expected trade-off between the values for $\gamma$ and $\beta$ to obtain fair-value contracts. A second observation is that if we keep participation rate $\gamma$ fixed, increasing the maturity $T$ needs to be compensated for by increasing the loading $\beta$. This brings us to the fact that this sort of trivial bonus policy has an unwanted side-effect for the minimum-value requirement or surrender feature.

It appears that for a fair-value contract with a constant participation rate the trade-off between this participation rate $\gamma$ and the loading $\beta$ becomes less favorable as the contract comes closer to maturity. That is, independent of the evolution of underlying asset or interest rates, the passing of time has an effect on the minimum-value requirement, which also functions as a surrender option. More precisely the contract becomes less favorable as time goes by, and therefore the likeliness that the policyholder will surrender the contract increases. This illustrates that the assumption of a constant participation rate is rather.

We now consider the case of a non-contingent or discretionary bonus policy. Remember that with such an approach the ‘fraction’ $1/w_i$ of the solvency reserve allocated to the contract at a reserve date $t_i$ is not contingent on the underlying assets or term structure of interest rates, but that we impose the condition that over the life of the contract the entire solvency
reserve $1/(1-\beta)$ is allocated back to the contract. With such a bonus policy the participation rate is reset at each reserve date $t_i$, which turns the $n$-period bonus policy into a series of $n$-one-period decisions.

Table 3: Value for $\gamma(1-\beta)$ implied by $w_i$

<table>
<thead>
<tr>
<th>$w_i$</th>
<th>$r(t_0) = 0.06$</th>
<th>$r(t_0) = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_G = 0.35$</td>
<td>$r_G = 0.025$</td>
</tr>
<tr>
<td>0.97</td>
<td>1.13</td>
<td>1.08</td>
</tr>
<tr>
<td>0.97</td>
<td>1.10</td>
<td>1.04</td>
</tr>
<tr>
<td>0.97</td>
<td>1.06</td>
<td>1.00</td>
</tr>
<tr>
<td>0.98</td>
<td>1.03</td>
<td>0.96</td>
</tr>
<tr>
<td>0.98</td>
<td>0.99</td>
<td>0.92</td>
</tr>
<tr>
<td>0.98</td>
<td>0.96</td>
<td>0.89</td>
</tr>
<tr>
<td>0.98</td>
<td>0.93</td>
<td>0.85</td>
</tr>
<tr>
<td>0.98</td>
<td>0.89</td>
<td>0.81</td>
</tr>
<tr>
<td>0.99</td>
<td>0.86</td>
<td>0.77</td>
</tr>
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<td>0.99</td>
<td>0.82</td>
<td>0.74</td>
</tr>
<tr>
<td>0.99</td>
<td>0.79</td>
<td>0.70</td>
</tr>
<tr>
<td>0.99</td>
<td>0.76</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Table 2.3 gives the value of the ’total participation rate’ $\gamma(1-\beta)$ such that the minimum-value requirement is met precisely, i.e. the surrender option is at the money for different values of $r_G$ and $r(t_0)$. The maturity $T$ of the contract is equal to 10 years. Further we have $r(t_0) = 0.06, a = 0.2, b = 0.1, \sigma_r = 0.03, \sigma_S = 0.2$ and $\rho = -0.2$. And as above, the fixed income component of the underlying assets is a rolling-over 8-year discount bond.

We see that, as one would expect, the higher the fraction $1/w_i$ that is allocated to the contract, the higher is the ’total participation rate’ $\gamma(1-\beta)$ over the next period. More interesting however is the observation that the total participation rate is sensitive to the value of $r(t_i)$, which leads to an interpretation of the solvency reserve $1/(1-\beta)$ illustrated in the Table 2.4

Table 2.4 is based on the same parameter values as Table 2.3 with the value of the loading $\beta$ set equal to 0.15, it gives the value for the participation rate $\gamma_i(\omega)$ at a reserve date $t_i$ implied by the fraction $1/w_i$ that is allocated to the contract at $t_i$.

Let us consider the contract with $r_G = 0.035$. Since the participation rate can not exceed one, we see that the fraction $1/w_i$ needs to be set below 1.0142 and 1.0267 when $r(t_i) = 0.06$.
Table 4: Value for $\gamma(1-\beta)$ implied by $w_i$ and $\beta$

<table>
<thead>
<tr>
<th>1/$w_i$</th>
<th>$r(t_0) = 0.06$</th>
<th>$r(t_0) = 0.03$</th>
<th>$r(t_0) = 0.06$</th>
<th>$r(t_0) = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_G = 0.35$</td>
<td>$r_G = 0.025$</td>
<td>$r_G = 0.35$</td>
<td>$r_G = 0.025$</td>
</tr>
<tr>
<td>1.0309</td>
<td>1.33</td>
<td>1.27</td>
<td>1.10</td>
<td>0.94</td>
</tr>
<tr>
<td>1.0288</td>
<td>1.29</td>
<td>1.22</td>
<td>1.04</td>
<td>0.88</td>
</tr>
<tr>
<td>1.0267</td>
<td>1.25</td>
<td>1.18</td>
<td>0.99</td>
<td>0.82</td>
</tr>
<tr>
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<td>0.93</td>
<td>0.76</td>
</tr>
<tr>
<td>1.0225</td>
<td>1.17</td>
<td>1.09</td>
<td>0.88</td>
<td>0.70</td>
</tr>
<tr>
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<td>0.82</td>
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</tr>
<tr>
<td>1.0183</td>
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<td>1.00</td>
<td>0.77</td>
<td>0.57</td>
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<td>0.55</td>
<td>0.33</td>
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<tr>
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<td>0.27</td>
</tr>
<tr>
<td>1.0060</td>
<td>0.85</td>
<td>0.73</td>
<td>0.45</td>
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</tbody>
</table>

or 0.03 respectively. On the other hand, if we assume that there is a minimum value for the participation rate of 0.8 then we see that when $r(t_i) = 0.03$ then $1/w_i$ needs be around 1.02 or higher. Since otherwise the implied value for the participation rate $\gamma$ would drop below the minimum level. This last observation illustrates how our model captures the fact that persistently low interest rates can significantly reduce the solvency reserves of life-insurance companies.

7 Conclusions

In this paper we have developed a model for life-insurance contracts that incorporates the main features of traditional with-profits contracts. It was demonstrated how in case of a multi-period contract the guaranteed return $r_G$ implies both a minimum-value requirement/surrender option and a periodic bonus feature. With respect to the bonus feature the authors assumed that a bonus is allocated by creating a so-called bonus contract, a common practise for with-profits contracts.

It was shown how the combination of the actuarial, non market-based, premium principle and a dynamic bonus policy leads to the existence of a solvency margin. Therefore, the choice of bonus policy comes down to choosing a strategy for allocating the solvency margin to the
contract. An interesting result is the fact that if a type of fair-value condition is met, the
discretion which insurance companies have in setting the bonus policy can actually be dealed
with in a contingent-claim pricing framework.

Possible paths for future research could be to include equityholders into the model, as in
Bryis and de Varenne (1997) or to allow for the possibility of early default such as in Grosen
and Jorgensen (2002). The observation periods of low interest rate levels reduce the solvency
margin could be a starting point for the latter suggestion.
Appendix A: Proof of Proposition 30

The contract with the minimum-value requirement/surrender feature is a Bermudan option with a time-varying strike price which can be seen as a compound option see Geske (1977) and Geske (1979). In order to prove Proposition 30 we need to derive the expressions for the strike prices $\{\Delta_i\}_{i=1}^n$.

**Proof:**

Step 1: $i = n - 2$

Since the pay-off of the contract at $t_n$ is given by:

$$D(t_n) + \gamma \frac{D(t_0)}{S(t_0)} [S(t_n) - D(t_n)],$$

the value alive of the Bermudan contract at $t_{n-1}$ is equal to:

$$\gamma \frac{D(t_0)}{S(t_0)} C(t_{n-1}, S(t_{n-1}), t_n, \Delta_n) + D(t_n) e^{-(t_n-t_{n-1}) r}. \tag{53}$$

Therefore the value alive at time $t_{n-2}$ is given by:

$$e^{-r(t_{n-1}-t_{n-2})} E^Q \left[ \max \left\{ \frac{D(t_0)}{S(t_0)} C(t_{n-1}, S(t_{n-1}), t_n, \Delta_n) + D(t_n) e^{-(t_n-t_{n-1}) r}, D(t_{n-1}) \right\} \right].$$

$$= e^{-r(t_{n-1}-t_{n-2})} D(t_{n-1}) + \frac{D(t_0)}{S(t_0)} E^Q \left[ C(t_{n-1}, S(t_{n-1}), t_n, \Delta_n) - \left( D(t_{n-1}) - D(t_n) e^{-(t_n-t_{n-1}) r} \right) \right] +$$

$$= \frac{D(t_0)}{S(t_0)} C(t_{n-2}, S(t_{n-2}), (t_k)_{k=1}^n, (\Delta_k)_{k=1}^n) + D(t_{n-1}) e^{-(t_{n-1}-t_{n-2}) r}. \tag{54}$$

That is:

$$\Delta_n = D(t_n)$$

$$\Delta_{n-1} = D(t_{n-1}) - D(t_n) e^{-(t_n-t_{n-1}) r}.$$

Observe that $\Delta_{n-1}$ is equal to:

$$\Delta_{n-1} = D(t_{n-1}) - D(t_n) e^{-(t_n-t_{n-1}) r}$$

$$= D(t_0) \left( e^{(t_{n-1}-t_0) r_G} - e^{(t_n-t_0) r_G} e^{-r(t_n-t_{n-1}) r} \right)$$

$$= D(t_0) e^{(t_{n-1}-t_0) r_G} \left( 1 - e^{(t_n-t_{n-1})(r_G-r)} \right),$$

33
Therefore, the minimum-reserve requirement or possibility to surrender the contract at \( t_{n-1} \) is redundant, and the the contract at \( t_{n-2} \) is a European call with maturity date \( t_n \).

Step 2: \( i + 1 \Rightarrow i \)

We need to prove that given that the proposition holds for a \((n - i - 1)\)-fold compound call, it holds for a \((n - i)\)-fold compound call. Or put differently, we need to demonstrate that if the proposition holds at a reserve date \( t_{i+1} \) of an \( n \)-period contract, it holds at \( t_i \).

We have that the value alive at \( t_i \) is given by:

\[
e^{-r(t_{i+1}-t_i)} \left[ E^Q \max \left\{ \frac{D(t_0)}{S(t_0)} C^{(n-i-1)}(t_{i+1}, S(t_{i+1}), (t_k)_{k=i+2}^n, (\Delta_k)_{k=i+2}^n) \right\} + D(t_{i+2})e^{-(t_{i+2}-t_{i+1})r} \right]
\]

\[
= \gamma \frac{D(t_0)}{S(t_0)} e^{-r(t_{i+1}-t_i)} E^Q \left[ C^{(n-i-1)}(t_{i+1}, S(t_{i+1}), (t_k)_{k=i+2}^n, (\Delta_k)_{k=i+2}^n) - \Delta_{i+1} \right] + \gamma e^{-r(t_{i+1}-t_i)} D(t_{i+1}),
\]

which is again a compound option, with \( \Delta_{i+1} \) given by:

\[
\Delta_{i+1} = D(t_{i+1}) - D(t_{i+2})e^{-(t_{i+2}-t_{i+1})r}.
\]

(55)

From which we again obtain that the the surrender option at \( t_{i+1} \) is redundant if \( r_G \geq r \).

\[
\boxed{\text{Appendix B: Proof of Propositions 31 and 32}}
\]

Observe that by combining Proposition 30 with the result below from Geske (1977) we obtain Proposition 31.

**Proposition 10** In an economy as described above, the value of an \( n \)-fold compound call option \( C^{(n)}(t_0, S(t_0); (t_k, K_k)_{k=1}^n) \) can be computed as follows:

**If** \( n = 1 \):

\[
C^{(1)}(t_0, S(t_0), t_1, K_1) = S(t_0)N(d_1) - e^{-(t_1-t_0)r}K_1N\left( d_1 - \sigma \sqrt{t_1-t_0} \right)
\]

(56)

**For** \( n > 1 \):
\[ C^{(n)}(t_0, S(t_0); (t_k, K_k)_{k=1}^n) = S(t_0)N_n \left( d_1^{(n)} + \sigma \sqrt{t_1 - t_0}, ..., d_n^{(n)} + \sigma \sqrt{t_n - t_0}; \Sigma^{(n,n)} \right) \]
\[ - \sum_{i=0}^{n-1} K_{n-i} e^{r(t_{n-i} - t_0)} N_{n-i} \left( d_1^{(n)}, ..., d_{n-i}^{(n)}; \Sigma^{(n,n-i)} \right), \] (57)

with:
\[ d_{n-i} = \frac{\ln \left( S(t_0)/\chi_{n-i} \right) + \left( r - \sigma^2/2 \right) (t_{n-i} - t_0)}{\sigma \sqrt{t_{n-i} - t_0}} \quad i = 0..n - 1 \] (58)

The for all \( i \in \{0..n - 1\} \) the value for \( \chi_{n-i} \) in equation 58 is given by the solution of the following equation:
\[ C^{(i)} \left( t_{n-i}, \chi_{n-i} \left( t_{n-j}, K_{n-j} \right)_{j=0}^{l-1} \right) = K_{n-i}. \] (59)

Here \( N_{n-i} \left( x_1, ..., x_{n-i}; \Sigma^{(n-n-i)} \right) \) is the multivariate normal probability determined by the \( (n-i) \) tuple \( (x_1, ..., x_{n-i}) \) and a multivariate normal distribution with a vector of means equal to the null vector and a covariance matrix \( \Sigma^{(n-n-i)} \). The elements \( \rho_{kl}^{(n,n)} \) of the matrix \( \Sigma^{(n,n)} \) are given by:
\[ \rho_{kl}^{(n,n)} = \sqrt{\frac{t_k - t_0}{t_l - t_0}} \quad \text{for} \quad k < l \]
\[ \rho_{kl}^{(n,n)} = 1 \quad \text{for} \quad k = l \]
\[ \rho_{kl}^{(n,n)} = \sqrt{\frac{t_l - t_0}{t_k - t_0}} \quad \text{for} \quad k > l. \] (60)

For all \( i = 1..n - 1 \) the matrix \( \Sigma^{(n-n-i)} \) is the submatrix of the \( n-i \) first rows and columns of matrix \( \Sigma^{(n,n)} \).

In a similar fashion we obtain Proposition 32 by combining Proposition 30 with the next result for the delta of a compound call option.

**Proposition 11** Under the Black-Scholes-Merton assumptions the delta of a compound option \( C^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^n) \) is given by:
\[ \delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^n) = N_n \left( d_1^{(n)} + \sigma \sqrt{t_1 - t_0}, ..., d_n^{(n)} + \sigma \sqrt{t_n - t_0}; \Sigma^{(n,n)} \right), \] (61)

with \( d_i^{(n)} \) \( (i = 1, ..n) \) Proposition 34.
One way to prove this proposition would be to explicitly compute the derivative of equation 57 with respect to $S(t_0)$. However, this is a very laborious approach. We will take a different approach, one that exploits the recursive nature of equation 57. Before we move on to give a proof of this proposition we first state a result on multivariate normal distributions which is a special case of a result in Lindset (2001)

**Proposition 12** For any integer $n > 0$ and positive real numbers $\{\Delta_i\}_{i=1}^n$, $\{t_i\}_{i=0}^n$, $r,s_0$ and $\sigma$, the following equality holds:

$$
\int_{\chi_1}^{+\infty} N_{n-1} \left( \frac{\left( \ln(s/\Delta_i) + (r - \sigma^2/2)(t_{i+1} - t_1) \right)}{\sigma \sqrt{t_{i+1} - t_1}} ; \Sigma^{(n-1,n-1)} \right) e^{\frac{1}{2} \frac{(\ln(s) - m)^2}{S^2}} ds
$$

$$
= N_n \left( \left( \frac{\ln(s_0/\Delta_i) + (r - \sigma^2/2)(t_i + 1 - t_1)}{\sigma \sqrt{t_i + 1 - t_1}} \right)^n ; \Sigma^{(n,n)} \right)
$$

(62)

with:

$$
m = s_0 + (r - \sigma^2/2)(t_1 - t_0)
$$

(63)

$$
S^2 = \sigma(t_1 - t_0)
$$

(64)

and $\Sigma^{(n,n)}$ and $\Sigma^{(n-1,n-1)}$ as in Theorem 2.

We are now ready to give the proof of Proposition XXXX.

**Proof :**

$n = 1$:

In this case equation 61 reduces to the well known result for the delta of a European call option.

$n > 1$:

Assume that equation 61 holds for $n - 1$. Note that under the assumption of the Black and Scholes model one has:

$$
C^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^n) = e^{-r(t_1-t_0)} E^Q \left[ \left[ C^{(n-1)}(t_1, S(t_1); (t_k, \Delta_k)_{k=2}^n) - \Delta_1 \right]_{+} S(t_0) \right]
$$
As such, equation 68 can be rewritten as:

$$\delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^{n}) = e^{-r(t_1-t_0)} \int_{\ln(x_1)}^{+\infty} \left( C^{(n-1)}(t_1, S(t_1); (t_k, \Delta_k)_{k=2}^{n}) - \Delta_1 \right) \frac{\partial}{\partial v} n(v; m, S^2) dv,$$

with $Q$ the (unique) equivalent martingale measure and $n(v; m, S^2)$ the normal density with mean $m$ and variance $S^2$, which are given by:

$$m = (r - \sigma^2/2)(t_1 - t_0) + \ln(S(t_0))$$

and:

$$S^2 = \sigma^2(t_1 - t_0).$$

From equation 65 one obtains:

$$\frac{\partial}{\partial v} n(v; m, S^2) = -\frac{v-m}{S^2} n(v; m, S^2).$$

As such, equation 68 can be rewritten as:

$$\delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^{n}) = e^{-r(t_1-t_0)} \frac{-e^{-r(t_1-t_0)}}{S(t_0)} \int_{\ln(x_1)}^{+\infty} \left( C^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^{n}) - \Delta_1 \right) \frac{\partial}{\partial v} n(v; m, S^2) dv$$

Partial integration leads to:

$$\delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^{n}) = e^{-r(t_1-t_0)} \frac{-e^{-r(t_1-t_0)}}{S(t_0)} \left( C^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^{n}) - \Delta_1 \right) \bigg|_{\ln(x_1)}^{+\infty} + e^{-r(t_1-t_0)} \frac{-e^{-r(t_1-t_0)}}{S(t_0)} \int_{\ln(x_1)}^{+\infty} \delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^{n}) n(v; m, S^2) dv.$$
The first term in the above equation is equal to zero since on the one hand:

\[ C^{(n-1)}(t_1, \chi_1; (t_k, \Delta_k)_{k=2}^n) - \Delta_1 = 0, \quad (72) \]

by definition of \( \chi_1 \); and on the other one has:

\[
\begin{align*}
\lim_{v \uparrow +\infty} \left( C^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) - \Delta_1 \right) n(v; m, S^2) &= \frac{1}{S\sqrt{2\pi}} \lim_{v \uparrow +\infty} \frac{C^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) - \Delta_1}{e^{\frac{1}{2}\left( \frac{v-m}{S} \right)^2}} \\
&= \frac{1}{S\sqrt{2\pi}} \lim_{v \uparrow +\infty} \frac{\delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n)}{e^{\frac{1}{2}\left( \frac{v-m}{S} \right)^2}} \\
&= 0 \quad (73)
\end{align*}
\]

The last equality holds because \( \delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) \leq 1 \) for all \( v \), and hence \( \lim_{v \uparrow +\infty} \delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) \leq 1 \), and at the same time: \( \lim_{v \uparrow +\infty} \frac{v-m}{S} e^{\frac{1}{2}\left( \frac{v-m}{S} \right)^2} = +\infty \).

As such one obtains:

\[
\delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^n) = \frac{e^{-r(t_1-t_0)}}{S(t_0)} \infty \delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) n(v; m, S^2)dv. \quad (74)
\]

Substituting the expressions for \( \delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) \) and \( n(v; m, S^2) \) leads to:

\[
\begin{align*}
\delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^n) &= \frac{e^{-r(t_1-t_0)}}{S(t_0)} \int_{\infty}^{\infty} \delta^{(n-1)}(t_1, v; (t_k, \Delta_k)_{k=2}^n) n(v; m, S^2)dv \\
&= \frac{e^{-r(t_1-t_0)}}{S(t_0)} \int_{\infty}^{\infty} N_{n-1} \left( \left( d_i^{(n-1)}(v) \right)_{i=1}^{n-1}; \Sigma^{(n-1,n-1)} \right) e^{-\frac{1}{2} \left( \frac{v-m}{S} \right)^2} dv \quad (75)
\end{align*}
\]

with:

\[
d_i^{(n-1)}(v) = \frac{v - \ln(\Delta_1) + (r - \sigma^2/2)(t_{i+1} - t_1)}{\sigma \sqrt{t_{i+1} - t_1}}
\]

The substitution \( v = \ln(s) \) and using equation 62 of Proposition 36 leads to:

\[
\delta^{(n)}(t_0, S(t_0); (t_k, \Delta_k)_{k=1}^n)
\]
\[
\begin{align*}
&= \frac{e^{-r(t_1-t_0)}}{S(t_0)} \times \\
&\int_{\chi_1}^{+\infty} sN_{n-1} \left( \left( \ln \left( \frac{s}{\Delta_i} \right) + \left( r + \frac{\sigma^2}{2} \right) (t_{i+1} - t_1) / \sigma \sqrt{t_{i+1} - t_1} \right) \right)^{n-1} \sigma \sqrt{t_{i+1} - t_1} \right)^n \Sigma_{(n,n)} ds \\
&= \frac{e^{-r(t_1-t_0)}}{S(t_0)} \left( e^{-r(t_1-t_0)} S(t_0) N_n \left( \left( \ln \left( \frac{S(t_0)}{\Delta_i} \right) + \left( r + \frac{\sigma^2}{2} \right) (t_{i+1} - t_1) / \sigma \sqrt{t_{i+1} - t_1} \right) \right)^n \Sigma_{(n,n)} \right) \\
&= N_n \left( \left( \ln \left( \frac{S(t_0)}{\Delta_i} \right) + \left( r + \frac{\sigma^2}{2} \right) (t_{i+1} - t_1) / \sigma \sqrt{t_{i+1} - t_1} \right) \right)^n \Sigma_{(n,n)} \right). \quad (76)
\end{align*}
\]
References


R. GESKE (1977), The Valuation of Corporate Liabilities as Compound Options, *Journal of Financial and Quantitative Analysis* 12, 541-552.


