Forward-Looking Interest Rate Rules and Hopf Bifurcations in Open Economies*

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Abstract

Empirical evidence shows that industrialized and emerging economies are following and have followed forward-looking interest rate rules as a manner to conduct monetary policy. In this paper we consider an open economy set-up with a sticky-price (non-traded) good and a flexible-price (traded) good, and pursue a global equilibrium analysis for these rules. We disentangle the conditions under which these rules may lead to aggregate instability by leading the economy to cyclical fluctuations that never converge asymptotically to the targeted equilibrium. We find that these conditions depend not only on how aggressive the rule responds to inflation and on the forward-looking character of the rule but also on structural parameters of the economy such as the degree of openness. Contrary to the recommendations from the optimal monetary policy literature in open and closed economies that suggests that governments should respond aggressively to the sticky price inflation in the rule we find that this is precisely what opens the possibility of cyclical dynamics due to a Hopf bifurcation. Responding to the flexible price inflation avoids the presence of these cyclical dynamics but makes the economy more prone to suffer from aggregate instability as a consequence of local multiple equilibria.

Keywords: Small Open Economy, Interest Rate Rules, Taylor Rules, Multiple Equilibria, and Endogenous Fluctuations.

JEL Classifications: E32, E52, F41

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*All errors remain ours. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the view of LUISS University or the Board of Governors of the Federal Reserve System or of any other person associated with the Federal Reserve System.

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1 Introduction

In recent years there has been a revival of theoretical literature aimed at understanding the macroeconomic consequences of implementing interest rate rules in closed and open economies models. This revival is partly explained by two reasons. First since the works by Taylor (1993) and Henderson and McKibbin (1993) it has become common to think about monetary policy in terms of these rules whereby the Central Bank maneuvers the nominal interest rate in response to inflation and output. Second there is empirical evidence suggesting that some industrialized and developing economies have followed in the past, and are following in the present, forward-looking (and contemporaneous) interest rate rules as a manner to conduct their monetary policy.

In this literature the study of interest rate rules, whose interest rate response coefficient to inflation is greater than one, has received particular attention. The reason is that these rules, also known as Taylor rules or active rules, have been claimed to be conducive to macroeconomic stability by alleviating inflationary pressures and by delivering a unique rational expectations equilibrium. On the contrary, rules whose interest rate response coefficient to inflation is less than one, also referred to as passive rules, have been claimed to induce aggregate instability in the economy by reinforcing inflationary pressures and by inducing endogenous self-fulfilling fluctuations.

Most of the closed and open economy literature that has emphasized the aforementioned benefits of active interest rate rules has also limited its analysis to characterize locally the equilibrium induced by these rules. In this sense this literature has concentrated on the behavior of endogenous variables that remain in a small neighborhood around the intended steady state and are expected to converge to it. An important exception to this approach in the closed economy literature is Benhabib et al. (2001,a,b, 2002, 2003). The reason

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3It is important to notice that this claim depends on the timing of the rule. In general this result is valid under local analysis for backward-looking rules, that is rules that respond to the past inflation, as shown by Carstrom and Fuerst (2001). However under some assumptions it may be also valid under forward-looking rules. See Bullard and Mitra (2002) and Carlstrom and Fuerst (2004).


is that they have pursued global equilibrium analyses and pointed out that there are rules that although
 guaranteeing a unique local equilibrium may in fact lead to aggregate macroeconomic instability by inducing
 non-convergent cyclical and chaotic dynamics.\footnote{A global analysis studies a larger set of equilibria in which endogenous variables may remain bounded but are never expected to return to their steady-state. It is in this sense that we use the term “non-convergent”.}

This observation and the fact that the open economy literature about interest rate rules has focused
on local equilibrium analysis is what motivates this paper.\footnote{An exception to this is Airaud and Zanna (2004) that in a flexible-price open economy set-up studies the possibility of cycles and chaos induced by interest rate rules.} Specifically we pursue a global equilibrium
analysis in order to explore the conditions under which (forward-looking) interest rate rules may induce
cyclical dynamics in open economies. To achieve this goal we use a continuous time microfounded small
open economy model with flexible-price traded goods and sticky-price non-traded goods. We assume the
government maneuvers the nominal interest rate in response to the weighted average of expected future
inflation rates. As a measure of inflation we consider the following three alternatives: the CPI-inflation, the
flexible price (traded) inflation or the sticky-price (non-traded) inflation. We find that forward–looking rules
may induce aggregate macroeconomic instability by embarking the economy in cyclical dynamics (that do
not converge to the steady state) due to a “Hopf bifurcation”.

Although at first glance this result may be perceived as a simple extension of the closed economy literature
results, our analysis of this result proves that there are some very important differences that policy makers in
open economies should consider. In fact the main contribution of this paper with respect to the closed (and
open) economy literature is two-fold. First, we show that the possibility of these global cyclical fluctuations
depend not only on the specification of the rule (the interest rate response coefficient to inflation and the
weights that the government puts on expected future inflations) but also on some structural parameters such
as the degree of openness of the economy and the degrees of imperfect competition and price stickiness in the
non-traded sector. Clearly the degree of openness of the economy, measured as the share of traded goods, is
an important characteristic of an open economy that may vary across economies. In particular we show that
when the measure of inflation coincides with the CPI-inflation, the more open the economy is, the lower the
interest rate response coefficient to inflation that is necessary for the appearance of non-convergent cycles in
the model. Through a simple calibration exercise, we provide a quantitative evaluation of how feasible and
relevant our analytically derived results are for the design of monetary policy.

Second by considering two goods with an important asymmetry about price-stickiness our paper contributes to the literature about which measure of inflation the central bank should respond to in order to avoid aggregate macroeconomic instability. Our results reveal that the local analysis and the global analysis differ on the recommendation about which measure of inflation should be included in the rule. Specifically the local analysis suggests that the measure of inflation should be the sticky-price (non-traded) inflation. The reason is that active rules with respect to this measure may guarantee locally equilibrium uniqueness. In contrast active rules with respect to the flexible-price (traded) inflation always lead to multiple equilibria. This result of responding to the sticky-price inflation agrees with the results of optimal monetary policy in the closed and open economy literature. In particular recent works in the open economy literature have suggested that governments in open economies should design interest rate rules satisfying two requirements. First the rule should be active. Second the measure of inflation included in the rule should be the domestic inflation or at least a measure of inflation of the sector that has (more) sticky prices and that filters out the transitory effects of exchange rate movements.

In contrast under the global equilibrium analysis we find that the possibility of endogenous non-convergent cycles due to a Hopf bifurcation arises only for rules that respond solely to the sticky-price (non-traded) inflation but not under rules that respond exclusively to the flexible-price (traded) inflation. In other words, contrary to the suggestions of the recent literature, a government that follows the policy advice of targeting the domestic (non-traded and sticky price) inflation may be actually end hurting the economy by opening the possibility of endogenous cycles that never converge asymptotically to the targeted equilibrium.

Benhabib et al. (2001b,2003) are important works related to this paper. In those works using a sticky-price closed economy model with money in the utility function and in the production function they pursue a global bifurcation analysis for contemporaneous and backward-looking rules. Under this set-up they find that the aforementioned rules may lead to cyclical dynamics due to a Hopf bifurcation. It is clear that in

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9 This result is also emphasized by Zanna (2003).
10 For the closed economy literature see the works by Aoki (2001) and Mankiw and Reis (2002) among others. These works suggest that the Central Bank should target the sticky-price inflation.
their works the assumption of money in the production function plays an important role for their results. Our results do not rely on this assumption since in our model money enters in a separate way in the utility function. Hence the existence of non-convergent cyclical fluctuations arises because of the forward-looking specification of the rule.\textsuperscript{12} To this extent the existence of non-convergent cyclical dynamics represents a severe and extreme case of policy-induced macroeconomic instability.

This paper is also related to Airaudo and Zanna (2004). Both are first attempts of the open economy literature to understand how (forward-looking) interest rate rules may lead to global cyclical endogenous fluctuations. The main difference is that in the present paper we consider a continuous-time open economy model with nominal rigidities and we focus on forward-looking rule whereas in Airaudo and Zanna (2004) we consider a discrete-time flexible-price small open economy and analyze both forward-looking and contemporaneous rules.

The remainder of this paper is organized as follows. Section 2 presents the set-up of the model with its main assumptions. Section 3 pursues the local determinacy analysis. Section 4 accomplishes the global bifurcation analysis. Finally Section 5 concludes.

2 The Model

2.1 The Household-Firm Unit

Consider a small open economy inhabited by a large number of identical household-firm unit. They have perfect foresight and live infinitely. Their individual preferences can be described by\textsuperscript{13}

\[
U_0 = \int_0^\infty \left[ A(c_{Tt}, c_{Nt}) + (1 - h_{Tt} - h_{Nt}(j)) + \chi \log(m_t) - \frac{\gamma}{2} \left( \frac{\hat{P}_{Nt}(j)}{\hat{P}_{Nt}(j)} - \bar{\pi}_t \right)^2 \right] e^{-\beta t} dt
\]

\[
A(c_{Tt}, c_{Nt}) = \alpha \log(c_{Tt}) + (1 - \alpha) \log(c_{Nt})
\]

where \(\alpha, \beta \in (0, 1)\), and \(\gamma, \chi > 0\); \(c_{Tt}\) and \(c_{Nt}\) denote the consumption of traded and non-traded goods respectively, \(h_{Tt}\) and \(h_{Nt}(j)\) are the labor allocated to the production of the traded good and to the \(j\)th

\textsuperscript{12} Note that it is possible to obtain cycles due to Hopf bifurcations in models with contemporaneous rules and money in the utility function. However one needs to assign a value of 0 to the subjective discount factor.

\textsuperscript{13} We use specific functional forms since this will simplify the analysis, allowing us to convey the main message of the paper.
variety of the non-traded good respectively and \( m_t \) refers to real money holdings \( \left( \frac{M_t}{E_t} \right) \). Equations (3) and (4) imply that the representative individual derives utility from consuming traded and non-traded goods, from not working and from the liquidity services of money.

To understand the last term in (3) we need to describe the production process. Labor is the only factor required for the production process of both goods and the technologies can be described by

\[
y_{Tt} = (h_{Tt})^{\theta_T} \quad \text{and} \quad y_{Nt}(j) = (h_{Nt}(j))^{\theta_N}
\]

where \( \theta_T, \theta_N \in (0, 1) \). We assume that there is perfect competition and flexible prices in the production of the traded good and imperfect competition and sticky prices in the production of the non-traded good. To model imperfect competition we assume that the representative agent can set the price \( P_{Nt}(j) \) for the \( j \)th variety of the non-traded good and that her sales are demand determined by the Dixit-Stiglitz constraint

\[
y_{Nt}(j) = Y_t^d \left( \frac{P_{Nt}(j)}{P_{Nt}} \right)^{\phi}
\]

where \( P_{Nt} \) is the economy-wide price level of the non-traded goods, \( Y_t^d \) is the aggregate demand for non-traded goods and \( \phi < -1 \). To model price stickiness we follow Rotemberg’s (1982) approach. Therefore we include some convex costs on the utility function (3) to capture the idea that the household-firm unit derives utility from hitting a target of own non-traded price change \( \left( \frac{P_{Nt}(j)}{P_{Nt}} \right) \) of the \( j \)th variety.\(^{14}\) That is the representative agent dislikes having her price of non-traded goods of the \( j \)th variety grow at a rate different from the steady-state non-traded good inflation rate, \( \bar{\pi}_N \).\(^{15}\)

The law of one price holds for the traded good and to simplify the analysis we normalize the foreign price of the traded good to one. Therefore, the domestic currency price of the traded good \( (P_{Tt}) \) is equal to the nominal exchange rate \( (E_t) \). That is \( P_{Tt} = E_t \). This simplification in tandem with the preferences aggregator described by equation (3) can be used to derive the consumer price index (CPI), \( p_t = \left( \frac{E_t}{\alpha(1-\alpha)} \right)^{\alpha} \). Using this and defining the nominal devaluation rate as \( \epsilon_t = E_t/E_t \), it is straightforward to derive the CPI inflation rate, \( \pi_t \), as a weighted average of the nominal depreciation rate, \( \epsilon_t \), and the inflation of the non-traded goods,\(^{14}\)Benhabib et al. (2001a,b) and Dupor (2001) also follow this approach to model price stickiness. An alternative approach follows Calvo (1983) and our results are invariant to this approach.

\(^{15}\)The upper bar refers to the steady state.
\[
\pi_{Nt} = \frac{P_{Nt}}{P_{Nt}}, \text{ that is } \\
\pi_t = \alpha \epsilon_t + (1 - \alpha) \pi_{Nt} \tag{5}
\]

It is important to notice that the weights in equation (5) depend on the share of traded goods, \(\alpha\). This share can be interpreted as a measure of the degree of openness of the economy in the present model. Moreover we define the real exchange rate \((e_t)\) as the ratio between the price of traded goods \((E_t)\) and the aggregate price of non-traded goods \((P_{Nt})\),

\[
e_t = \frac{E_t}{P_{Nt}} \tag{6}
\]

From this definition it is straightforward to deduce that

\[
\frac{\dot{e}_t}{e_t} = \epsilon_t - \pi_{Nt} \tag{7}
\]

The representative household-firm unit can invest in two types of interest-bearing assets: domestic bonds issued by the government, \(A_t\), that pay a nominal interest rate, \(R_t\); and foreign currency denominated bonds, \(b_t\), that pay an interest rate, \(r_t\). The real values of these assets will be denoted by \(a_t = A_t/E_t\) and \(b_t\), respectively. Then the representative agent’s instantaneous budget constraint can be written as follows

\[
\dot{b}_t = r_t b_t + \frac{c_{Nt}}{e_t} + \epsilon_t (m_t + a_t) + (\dot{m}_t + \dot{a}_t) - R_t a_t - \tau_t - y_{Tt} - \frac{P_{Nt}(j) y_{Nt}(j)}{P_{Nt}} \frac{e_t}{e_t} \tag{8}
\]

where \(\tau_t\) denotes lump-sum transfers from the government. Equation (8) says that the accumulation of foreign bonds is equal to the difference of the agent’s expenditures and her disposable income. Her expenditures consist of interest paid on foreign debt, her consumption of traded and non-traded goods, and her holdings of money and domestic bond balances, eroded by domestic currency depreciation. Her income is determined by the interests received by domestic bonds, the transfers from the government, and her income from producing and selling the traded good and the \(jth\) variety of the non-traded good.

Finally the representative household-firm unit is also subject to a Non-Ponzi game condition of the form

\[
\lim_{t \to \infty} e^{-\beta t} (m_t + a_t + b_t) \geq 0
\]

The problem of the agent is reduced to choose \(c_{Tt}, c_{Nt}, h_{Tt}, h_{Nt}(j), m_t, a_t, b_t\) and \(P_{Nt}(j)\) in order to maximize (??) subject to (??), (??), (??), (??) and the Non-Ponzi game condition, given \(b_0, a_0, m_0, \ldots\)
$P_{N0}(j)$, $\bar{\pi}_N$ and the time paths for $r$, $R_t$, $\epsilon_t$, $Y_t^d$, $P_{Nt}$ and $\tau_t$. The first order conditions associated with this optimization problem correspond to (9) and

\[ \frac{\alpha}{c_T} = \lambda \]  

\[ \frac{\alpha c_N}{(1 - \alpha)c_T} = e \]  

\[ \left( \frac{P_N(j)}{P_N} \right) \left( \frac{1}{e} - \frac{\mu}{\lambda} \right) \theta_N (h_N(j))^{(\theta_N - 1)} = \frac{1}{\lambda} = \theta_T h_T^{(\theta_T - 1)} \]  

\[ m = \chi c_T \frac{\alpha}{R} \]  

\[ \dot{\lambda} = \lambda(\beta - r) \]  

\[ R - \epsilon = r \]  

\[ \pi_N(j) = r(\pi_N(j) - \bar{\pi}_N) - \frac{\lambda}{\gamma} \frac{P_N(j) h_N^{\theta_N}(j)}{P_N} \left( \frac{1}{\lambda} \right) - \frac{\mu}{\gamma} \frac{P_N(j) Y_t^d d}{P_N} \]  

\[ \lim_{t \to \infty} e^{-\beta t} (m + a + b) = 0 \]

where $\lambda$ is the co-state variable or in economic terms the shadow price of wealth, $\mu$ is the multiplier associated with the demand constraint (11) and $\pi_N(j) = \frac{\dot{P}_N(j)}{P_N(j)}$. From now on we will focus on a symmetric equilibrium for which $P_N(j) = P_N$ and $h_N(j) = h_N$. We proceed giving an interpretation to the first order conditions. Equation (9) equates the marginal utility of traded goods to the marginal utility of wealth. Equation (10) implies that the marginal rate of substitution between traded and non-traded goods is equal to their relative price or real exchange rate. Equation (11) equalizes the marginal revenue products of labor among sectors whereas equation (12) describes the demand for real balances of money as an increasing function of the consumption of traded goods and a decreasing function of the nominal interest rate of the domestic
bonds offered by the government. Equation (??) corresponds to the Euler equation for the shadow price of wealth and equation (??) can be seen as an Uncovered Interest Parity (UIP) condition that equals the real returns from the domestic bond and the foreign bond.

2.2 The Government

The government issues two nominal liabilities: money, $M^g$, and a domestic bond, $A^g$, that pays a nominal interest rate $R$. The real values of these nominal variables are denoted by $m^g$ and $a^g$, respectively. It receives revenues from seigniorage and uses them to make lump-sum transfers to households, $\tau$, and to pay interest on its debt ($Ra^g$). Under these assumptions the government budget constraint can be written as follows

$$\dot{m}^g + \dot{a}^g + \epsilon(m^g + a^g) = Ra^g + \tau \tag{17}$$

The fiscal policy is Ricardian. That is the government picks the path of $\tau$ satisfying the intertemporal version of (??) in conjunction with the transversality condition,

$$\lim_{t \to \infty} (m^g + a^g) \exp \left( - \int_0^t (R - \epsilon) ds \right) = 0 \tag{18}$$

Finally we define the monetary policy as a forward-looking interest-rate feedback rule whereby the government sets the nominal interest rate as continuous and increasing function of the weighted average of expected future rates of the CPI-inflation ($\pi_f$)$^{17}$

$$R = \rho(\pi_f) = \bar{R} + \rho_\pi (\pi_f - \bar{\pi}) \quad \rho_\pi > 0 \tag{19}$$

where

$$\pi_f = k \int_t^\infty \pi(s) e^{-k(s-t)} ds \quad k > 0 \tag{20}$$

and $k$ measures the weight that the monetary authority puts on inflation rates of the future.$^{18}$ If $k$ is small then the central bank puts a large weight on inflation rates of the distant future.

$^{16}$Note that the government has no access to foreign bonds. This assumption simplifies the model and it does not have serious implications for our results if the interest is in analyzing cases in which the exchange rate is flexible.

$^{17}$Below we also consider the cases in which the measure of inflation corresponds to either the traded (flexible-price) inflation rate or the non-traded (sticky-price) inflation rate.

$^{18}$Henceforth when we use the term “forward-looking” rule it should be understood that the rule responds to the weighted average of expected future inflation.
Taking the derivative with respect to time to both sides of (21) and applying Leibniz’s rule we can find a differential equation that is useful to describe the dynamics of $\pi_f$

$$\dot{\pi}_f = k(\pi_f - \pi) \quad (21)$$

2.3 The Current Account and Capital Markets

The equation that describes the behavior of the current account in this model can be derived adding up equations (21) and (22) and using the equilibrium symmetry conditions $P_N(j) = P_N$, $h_N(j) = h_N$, and the equilibrium conditions for the non-traded good market, $y_N = h^*_N = c_N$, the money market, $m = m^g$, and the domestic bond market, $a = a^g$. Then we obtain

$$\dot{b} = rb + c_T - y_T \quad (22)$$

We introduce imperfect capital markets using an ad-hoc upward-sloping supply curve of funds on the world capital market. In this sense the small borrowing economy faces a world interest rate, $r$, that increases with the level of external debt. One of the simplest formulations of the upward-sloping supply curve was stated by Bardhan (1967) in which the cost of debt increases with the absolute level of foreign debt. We follow this approach and assume that

$$r = r^* + \eta(b - \bar{b}) \quad (23)$$

where $r^*$ is the risk free international interest rate, that is considered constant, and $\eta(b - \bar{b})$ is the country-specific risk premium. This premium increases when the stock of the debt issued by the country ($b$) is above its long run level ($\bar{b}$). To simplify the analysis we also suppose that the domestic nation operates in a range in which rationing from the international capital markets is not possible.

The assumption of imperfect capital markets seems to be a more realistic assumption for developing economies, since international capital markets are likely to react to their perception of a country’s ability to repay its debt. More importantly introducing this ad-hoc upward-sloping supply curve of funds is a way to solve the “zero-root” problem of small open economy models.

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19 The intuition that is behind of this ad-hoc structure is the following. As the external debt grows so does the risk of default and in order to compensate the lenders for this risk it is necessary to pay them a premium over the safe lending rate or risk free rate, $r^*$.

2.4 A Perfect Foresight Equilibrium

To give the definition of the perfect foresight equilibrium in this model, we can reduce the model in the following way. Using equations (23) and (24) and assuming that $\beta = r^*$ we can derive the Euler equation for consumption of traded goods.

$$\frac{\dot{c}_T}{c_T} = \eta(b - \bar{b})$$

(24)

Similarly using equations (22), (23), (25), (26) and (27) together with the assumption $\beta = r^*$, we can derive an Euler equation for the consumption of non-traded goods,

$$\frac{\dot{c}_N}{c_N} = R - \pi_N - r^*$$

(25)

On the other hand utilizing equations (20), (21), (22), (23) and (24), and the equilibrium symmetry conditions

$$P_N(j) = P_N, \ h_N(j) = h_N, \ \pi_N(j) = \pi_N(j),$$

altogether with the equilibrium condition for the non-traded good $(y_N = h_N^\beta = c_N)$ and $\beta = r^*$, we can obtain

$$\hat{\pi}_N = r^*(\pi_N - \pi^*_N) - \frac{(1 + \phi)(1 - \alpha)}{\gamma} + \frac{\phi}{\gamma\theta_N}(c_N)^{\frac{1}{\theta_N}}$$

(26)

where $\phi < -1$. This equation corresponds to a augmented Phillips equation for non-traded goods inflation.\(^{21}\)

In addition using equations (25), (26) and (27) we can rewrite the current account equation in (22) as

$$\dot{b} = [r^* + \eta(b - \bar{b})] b + c_T - \left(\frac{c_T}{\alpha\theta_T}\right)^{\frac{\theta_T}{\theta_T}}$$

(27)

Finally utilizing equations (22) and (27) we can rewrite the UIP condition as

$$R - \epsilon = r^* + \eta(b - \bar{b})$$

(28)

**Definition 1** Given $b_0$, $\bar{\pi}_N$, $\bar{b}$, and $r^*$ and under the assumption that the fiscal regime is Ricardian, a Perfect Foresight and Symmetric Equilibrium is defined as a set of sequences \(\{c_T, c_N, \pi_N, \pi, \epsilon, \pi^*_N, R, b\}\) satisfying: a)

\(^{21}\)We would have derived a similar equation if we had followed Calvo (1983). Note that iterating forward this equation we obtain

$$\pi_{Nt} = \int_t^\infty e^{-r(s-t)} \frac{-\phi}{\gamma} \left(c_{Ns}^{\frac{1}{\theta_N}} - (\bar{y}_N)^{\frac{1}{\theta_N}}\right) ds$$

where $(\bar{y}_N)^{\frac{1}{\theta_N}} = \frac{(1-\alpha)(1+\phi)\theta_N}{\gamma}$. This forward-looking expression describes the current inflation of non-traded goods as a function of the expected future excesses of demand in the non-traded goods market. Therefore as in Calvo (1983), the change of the non-traded goods inflation rate is a negative function of the excess of demand, given that $\phi < -1$. 

11
The Euler equation for consumption of traded goods, equation (??), b) The Euler equation for consumption of non-traded goods, equation (??), c) The augmented Phillips equation for non-traded goods inflation, equation (??), d) The definition of the CPI-inflation, equation (??), e) The UIP condition, equation (??), f) The interest rate feedback rule, equation (??), g) Equation (??) and h) The current account equation (??).

With the equations in Definition 1 it is possible to find the steady state in this economy. Given \( \bar{\pi}_N \) and \( \bar{b} \), the steady state in this economy is given by

\[
\bar{\pi}_N = \bar{\pi} = \bar{\epsilon} = \bar{\pi}_f \quad \bar{R} = r^* + \bar{c} \quad (\bar{c}_N)^{\frac{\phi}{\gamma}} = \frac{(1 - \alpha)(1 + \phi)\theta_N}{\phi} \quad r^*\bar{b} + \bar{c}_T = \left( \frac{\bar{c}_T}{\alpha \theta_T} \right)^{\frac{\sigma_T}{\gamma - 1}}
\]

(29)

In addition given the equilibrium set of sequences \{c_T, c_N, \pi_N, \pi, \epsilon, \pi_f, R, b\} then the corresponding sequences \{\lambda, \epsilon, m, \alpha, h_T, h_N, \mu, h_N\}, are uniquely determined by equations (??), (??), (??), (??), (??), (??) and the equilibrium condition for non-traded goods.

3 The Local Equilibrium Analysis

In order to accomplish the determinacy of equilibrium analysis we reduce the model further. In fact using all the equations of Definition 1 and the steady state equations (??) we obtain

\[
\frac{\dot{c}_T}{c_T} = \eta(b - \tilde{b})
\]

\[
b = [r^* + \eta(b - \tilde{b})]b + c_T - \left( \frac{c_T}{\alpha \theta_T} \right)^{\frac{\sigma_T}{\gamma - 1}}
\]

\[
\dot{\pi}_f = k(1 - \alpha \rho_N)(\pi_f - \bar{\pi}) + k\alpha [\bar{R} + \eta(b - \tilde{b})] - k(1 - \alpha)(\pi_N - \bar{\pi})
\]

(30)

\[
\dot{\pi}_N = r^*(\pi_N - \bar{\pi}) - \frac{(1 + \phi)(1 - \alpha)}{\gamma} + \frac{\phi}{\gamma \theta_N} (c_N)^{\frac{1}{\theta_N}}
\]

\[
\frac{\dot{c}_N}{c_N} = \rho_N(\pi_f - \bar{\pi}) - (\pi_N - \bar{\pi})
\]
An important feature of system (??) that will allow us to derive all the results analytically is that the specification of the interest rate rule does not affect neither the Euler equation of the consumption of traded goods nor the current account equation. This is just a consequence of our separable in all arguments utility function. In addition, to facilitate the determinacy of equilibrium analysis we state and prove the following Lemma.

**Lemma 1** Define \( \Omega \) and \( \rho^H \) as
\[
\Omega = \frac{-(1+\phi)}{\delta_N \gamma k r^*} > 0 \quad \text{and} \quad \rho^H = \frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r^*}{2\alpha k} - \frac{\sqrt{\Phi}}{2\alpha^2 k},
\]
where \( \Phi = \left[\Omega(1-\alpha)^2 k + \alpha r^*\right]^2 + 4\Omega \kappa(1-\alpha) [(1-\alpha) k + \alpha r^*] \) and assume that \( \Omega < 1 \) then

a) \( 1 < \rho^H < \frac{1-\Omega}{\alpha} + \Omega \) and \( \rho^H < \frac{1}{\alpha} \) for \( \alpha \in (0, 1) \),

b) \( \lim_{\alpha \to 0} \rho^H = 1 \) and \( \lim_{\alpha \to 0} \rho^H = \frac{1}{1+\Omega} + \frac{r^*(1-\Omega)}{1\kappa} > 1 \), and
c) \( \frac{\partial \rho^H}{\partial \alpha} < 0, \ \frac{\partial \rho^H}{\partial k} > 0, \ \frac{\partial \rho^H}{\partial \phi} > 0, \) and \( \frac{\partial \rho^H}{\partial \gamma} > 0 \) given that \( 1 \leq \rho^H < \frac{1}{\alpha} \).

**Proof.** See Appendix. ■

Using (??) and Lemma 1 we proceed to pursue the determinacy of equilibrium analysis for interest rate rules that are defined in terms of the weighted average of the expected future CPI-inflation rates, \( R = \rho(\pi_f) \).

The motivation of this analysis can arise from empirical works like Orphanides (1997), Clarida et al. (2000) and Corbo (2000). They argue that the central bank behavior in industrialized economies and emerging economies has been primarily forward-looking.

**Proposition 1** Let \( \Omega \) and \( \rho^H \) be defined as in Lemma 1 and assume that the central bank follows a forward-looking rule given by (??). Assume that \( \Omega < 1 \) then

a) If \( \rho_\pi < 1 \) (a passive rule) then there exists a continuum of perfect foresight equilibria in which, \( \{c_T, e_N, b, \pi_f, \pi_N\} \) converge asymptotically to the steady state.

b) If \( 1 < \rho_\pi < \rho^H \) (an active rule) then there exists a unique perfect foresight equilibrium in which \( \{c_T, e_N, b, \pi_f, \pi_N\} \) converge asymptotically to the steady state.

c) If \( \rho^H < \rho_\pi \) (an active rule) then there exists a continuum of perfect foresight equilibria in which \( \{c_T, e_N, b, \pi_f, \pi_N\} \) converge asymptotically to the steady state.\(^{22}\)

**Proof.** See Appendix. ■

\(^{22}\)This is an active rule because \( \rho^H > 1 \) as we stated in Lemma 1.
Proposition 1 states that conditions under which forward-looking interest rate rules generate aggregate instability in the economy by inducing local multiple equilibria depend non only on the interest rate response coefficient $\rho_\pi$, but also on structural characteristics of the economy that determine the threshold $\rho^H$. In fact using Lemma 1 we can see that this threshold depend negatively on the degree of openness of the economy, $\alpha$, (share of traded goods) and positively on the degrees of imperfect competition, $\phi$, and price-stickiness, $\gamma$, in the non-traded sector. We can use Lemma 1 and Proposition 1 to illustrate how keeping the rest constant the interaction between the specification of the rule and the degree of openness of the economy may affect the determinacy of equilibrium.\footnote{Similar analysis can be done for the degrees of imperfect competition, $\phi$, and price-stickiness, $\gamma$, in the non-traded sector.} Figure 1 accomplishes this goal by illustrating that the more open the economy (the higher $\alpha$), the more likely that an active forward-looking rule with $\rho_\pi > 1$ will induce multiple equilibria (real indeterminacy).\footnote{From now on we will use the terms “multiple equilibria” and “real indeterminacy” (a “unique equilibrium” and “real determinacy”) interchangeably. By real indeterminacy we mean a situation in which the behavior of one or more (real) variables of the model is not pinned down by the model. This situation implies that there are multiple equilibria and opens the possibility of the existence of sunspot equilibria.} Figure 1 shows how the local determinacy of equilibrium varies with respect to the CPI-inflation coefficient ($\rho_\pi$) and the share of traded goods ($\alpha$). “I” stands for real indeterminacy (multiple equilibria) and “D” stands for real determinacy (a unique equilibrium). Figure

It is important to understand how the forward-looking character of the rule drives some of the previous results. Specifically note that the forward-looking character of the rule is to some extent linked to $k$, the weight that the central bank puts on future CPI-inflation rates, in the definition of $\pi_f$. The smaller $k$, then the greater the weight the central bank puts on inflation rates of the distant future. That is, the smaller $k$, the more forward-looking the monetary authority becomes. Using the definition of $\rho^H$ in Lemma 1 it is simple to see that as $k \to \infty$ then $\rho^H \to \frac{1}{\alpha}$ and from Figure 1 we can infer that the area of real determinacy increases. In other words it becomes more likely that an active interest rate rule guarantees a unique equilibrium for any share of traded goods.\footnote{Note that as $k \to \infty$ then from (??) we can see that $\pi_f \to \pi$. This is the case analyzed in Zanna (2003).}

In addition, the possibility that a forward-looking rule guarantees a unique equilibrium depends not only on the absolute magnitude of $k$ but also on its relative value with respect to $\frac{- (1 + \phi)}{\gamma r_k f}$. In other words we may interpret that a central bank is “excessively forward-looking” if $k$ is small enough such that $\Omega = \frac{- (1 + \phi)}{\gamma r_k f} > 1$.\footnote{Note that as $k \to \infty$ then from (??) we can see that $\pi_f \to \pi$. This is the case analyzed in Zanna (2003).}
The reason is that the assumption $\Omega < 1$ in Proposition 1 is not innocuous. In particular an “excessively forward-looking” central bank that satisfies $\Omega = \frac{-(1+\phi)}{\theta_N - \theta_L} > 1$ will induce multiple equilibria regardless of the magnitude of the interest rate response coefficient, $\rho_\pi$, in the rule. The following Proposition formalizes this statement.

**Proposition 2** Define $\Omega$ as $\Omega = \frac{-(1+\phi)}{\theta_N - \theta_L}$ and assume that the central bank follows a forward-looking rule given by (??). Assume that $\Omega > 1$ then there exists a continuum of perfect foresight equilibria in which $\{c_T, c_N, b, \pi_f, \pi_N\}$ converge asymptotically to the steady state for any $\rho_\pi > 0$.

**Proof.** See Appendix. ■

The results in Figure 1 have an interesting interpretation in terms of the measure of inflation that the central bank responds to in the interest rate rule. In fact most of the previous results will still hold if the monetary authority instead of responding to the weighted average of expected future CPI-inflation rate, responds to the weighted average of the expected future core inflation rates; where the core inflation is defined as $\pi^* = w\epsilon + (1-w)\pi_N$ and $w \in [0,1]$ is just a weight determined by the government. To see this use the definition of $\pi^*$ and define $\pi_f^*$ as

$$\pi_f^* = k \int_t^\infty \pi^*(s)e^{-k(s-t)}ds = wk \int_t^\infty \epsilon(s)e^{-k(s-t)}ds + (1-w)k \int_t^\infty \pi_Nf(s)e^{-k(s-t)}ds$$

then

$$\pi_f^* = w\epsilon_f + (1-w)\pi_{Nf}$$ (31)

Hence the weighted average of expected future core inflation rates, $\pi_f^*$, is equal to the convex combination of the weighted average of expected future traded inflation rates, $\epsilon_f$, and the weighted average of expected future non-traded inflation rates, $\pi_{Nf}$. On the other hand using (??) and (??) we can derive that

$$\pi_f = \alpha\epsilon_f + (1-\alpha)\pi_{Nf}$$ (32)

The main difference between (??) and (??) is that $\alpha \in (0,1)$ is a structural parameter determined by the consumer-price index while $w \in [0,1]$ is a “policy” parameter determined by the government. Therefore if in Figure 1 we reinterpret $\alpha$ as $w$, we can infer the following. When the central bank responds exclusively
to the weighted average of the expected future traded good (flexible-price) inflation rates \( \epsilon_f \), that is when \( w \rightarrow 1 \), then the rule will always cause aggregate instability in the economy by generating multiple equilibria. On the other hand if the central bank responds actively and solely to the weighted average of the expected future non-traded good (sticky-price) inflation rates \( \pi_{Nf} \), that is when \( w \rightarrow 0 \), then the rule may guarantee a unique equilibrium.

The previous argument suggests that in the context of forward-looking rules that the measure of inflation that the central bank should respond to is the sticky-price inflation.\(^\text{26}\) This result coincides to some extent with the proposals of the optimal monetary policy literature such as Aoki (2001), Clarida et al. (2001) and Mankiw and Reis (2002). To formalize these ideas we pursue the determinacy of equilibrium analysis of two rules that differ in the measure of inflation that is included in the specification. In the first one the rule responds exclusively to the weighted average of the expected future traded good (flexible-price) inflation rates \( \epsilon_f \). That is,

\[
R = \bar{R} + \rho_\pi (\epsilon_f - \bar{\pi}) \quad \rho_\pi > 0
\]

where \( \epsilon_f = k \int_0^\infty e^{-k(s-t)} ds \) with \( k > 0 \) and \( \dot{\epsilon}_f = k(\epsilon_f - \epsilon) \). The following proposition summarizes the results for this rule.

**Proposition 3** Define \( \Omega \) as \( \Omega = -\frac{(1 + \phi)}{\theta N^{\gamma/r}} \) and assume that the central bank follows a forward-looking rule given by (??), then there exists a continuum of perfect foresight equilibria in which \( \{c_T, c_N, b, \epsilon_f, \pi_N\} \) converge asymptotically to the steady state

**Proof.** See Appendix. ■

Next, we pursue the determinacy of equilibrium analysis for a rule whereby the central bank responds actively and solely to the weighted average of the expected future non-traded good (sticky-price) inflation rates \( \pi_{Nf} \). In other words we define the rule as

\[
R = \bar{R} + \rho_\pi (\pi_{Nf} - \bar{\pi}) \quad \rho_\pi > 0
\]

where \( \pi_{Nf} = k \int_t^\infty \pi_N(s)e^{-k(s-t)} ds \) with \( k > 0 \) and \( \dot{\pi}_{Nf} = k(\pi_{Nf} - \pi_N) \). To facilitate the analysis of this system we first state and prove the following Lemma.

---

\(^{26}\)See Zanna (2003b).
Lemma 2 Define Ω and ρ as Ω = \(-\frac{(1+\phi)}{\theta N k}\) > 0 and ρ = \(\frac{1}{\Omega(1-\alpha)} \left\{ 1 + \frac{r(1-\Omega)}{\Omega(1-\alpha)} \right\}\) respectively and assume that Ω < 1 then a) 1 < ρ for α ∈ (0, 1), b) \(\lim_{\alpha \to 1} \rho = \infty\) and \(\lim_{\alpha \to 0} \rho = 1\) and c) \(\frac{\partial \rho}{\partial \alpha} > 0\), \(\frac{\partial \rho}{\partial k} > 0\), \(\frac{\partial \rho}{\partial \phi} > 0\), and \(\frac{\partial \rho}{\partial \gamma} > 0\).

Proof. See Appendix. ■

Using this Lemma and system (??) we can characterize locally the equilibrium under the aforementioned rule in the following proposition.

Proposition 4 Let Ω and ρ be defined as in Lemma 2 and assume that the central bank follows a forward-looking rule given by (??) and that Ω < 1.

a) If ρ < 1 (a passive rule) then there exists a continuum of perfect foresight equilibria in which, \(\{c_T, c_N, b, \pi_{Nf}, \pi_N\}\) converge asymptotically to the steady state.

b) If 1 < ρ < ρ (an active rule) then there exists a unique perfect foresight equilibrium in which \(\{c_T, c_N, b, \pi_{Nf}, \pi_N\}\) converge asymptotically to the steady state.

c) If ρ < ρ (an active rule) then there exists a continuum of perfect foresight equilibria in which \(\{c_T, c_N, b, \pi_{Nf}, \pi_N\}\) converge asymptotically to the steady state27.

Proof. See Appendix. ■

From Proposition 4 it is clear that the determinacy of equilibrium conditions for rules that respond to the weighted average of expected future non-traded goods inflation rate lead to multiple equilibria do not simply depend on the response coefficient ρ. On the contrary some of the structural parameters of the model play a fundamental role in the determinacy of equilibrium.28 In essence all the parameters that affect the threshold ρ are relevant for the analysis. In order for the rule to guarantee a unique multiple equilibria two conditions must be satisfied. The first one implies that the rule should be active 1 < ρ; the second one requires that the interest rate coefficient be less than a threshold ρ.

27This is an active rule because ρ > 1 as we stated in Lemma 2.
28Note also the importance of the assumption Ω < 1. As before it can be shown that if Ω(1 - α) > 1, then regardless of the interest rate response coefficient to inflation, the rule will always induce multiple equilibria.
We can proceed by studying how important the share of traded goods (degree of openness of the economy) is in the analysis. To accomplish this we can use Lemma 2 and Proposition 4 to construct Figure 2 under the assumption that $\Omega < 1$. Interestingly this figure shows that as more open the economy is (the larger $\alpha$ is), the more likely is that an active forward-looking rule with respect to the non-traded (sticky-price) goods will guarantee a unique equilibrium. This result has another interpretation. It suggests that as the share of flexible-price (traded) goods increases it becomes more important for the central bank to respond actively to the sticky-price (non-traded) goods inflation in order to avoid aggregate instability induced by multiple self-fulfilling equilibria.

This figure shows how the local determinacy of equilibrium varies with respect to the non–traded goods inflation coefficient ($\rho_n$) and the share of traded goods ($\alpha$). “I” stands for real indeterminacy (multiple equilibria) and “D” stands for real determinacy (a unique equilibrium). Figure

Summarizing, the results from Propositions 1, 2 and 3 suggest that conditions under which forward-looking rules lead to aggregate instability in open economies depend not only on the specification of the rule but also on some structural parameters such as the share of traded goods (degree of openness of the economy). From this local determinacy of equilibrium analysis it is clear that a central bank that follows a forward-looking rule should never respond to the traded (flexible-price) goods inflation. Only rules that respond to the non-traded (sticky-price) goods inflation may guarantee a unique equilibrium. In fact the more open the economy is, the stronger the incentives are to respond to this particular measure of inflation in order to avoid destabilizing the economy.\(^{29}\)

4 The Global Bifurcation Analysis

We proceed to pursue a global equilibrium analysis in order to evaluate if the aforementioned results still hold. So far we have focused on perfect foresight equilibria in which $\{c_T, c_N, b, \pi_f, \pi_N\}$ converge asymptotically to their steady state. However we want to study if there are perfect foresight equilibria in which the economy converges to a deterministic cycle.\(^ {30}\) If this is the case we want to explore the conditions under which this

\(^{29}\)This is just a restatement of the results in Zanna (2003) in the context of forward-looking rules.

\(^{30}\)In this case the equilibria are still bounded but they never converge to the steady state.
possibility exists. We will show that this possibility may arise because of the presence of a Hopf bifurcation. Technically such a bifurcation may appear if under some structural conditions the system (??) has a pair of imaginary eigenvalues with zero real parts and no other eigenvalues with zero real parts.31

We start by exploring the possibility of the aforementioned bifurcations under a forward-looking rule that responds to the CPI-inflation like the one defined in (??).

Proposition 5 Let Ω and ρH be defined as in Lemma 1 and assume that the central bank follows a forward-looking rule such that $R = \bar{R} + \rho_\pi (\pi_f - \bar{\pi})$ with $\rho_\pi > 0$ and $\pi_f = k \int^\infty_t \pi(s)e^{-k(s-t)}ds$ with $k > 0$. Assume that $\Omega < 1$ then there exists a critical value $\rho_\pi = \rho^H$ such that the dynamic system described by (??) displays a Hopf bifurcation.

Proof. See Appendix. ■

Proposition 5 points out that a forward-looking rule with respect to the CPI-inflation rate may induce non-convergent cyclical dynamics and therefore generate aggregate instability in the economy. In particular it is important to notice that the critical value for the appearance of these endogenous fluctuation is defined in terms of the interest rate response coefficient $\rho_\pi = \rho^H$. Part c) ofLemma 1 points out that this critical value $\rho_\pi = \rho^H$ is in fact related to structural parameters of the economy such as the share of traded goods, $\alpha$, and the degrees of monopolistic competition, $\phi$, and price stickiness, $\gamma$, in the non-traded sector, as well as to the weight that the monetary authority puts on expected future CPI-inflation rates in the rule, $k$. For instance part c) of Lemma 1 and Figure 1 illustrate that the greater $\alpha$ is (the more open the economy is), the smaller the critical interest rate response coefficient ($\rho_\pi = \rho^H$) that is necessary for the appearance of cyclical fluctuations due to Hopf bifurcations. Similarly from the aforementioned Lemma we can deduce that the smaller $k$ is (the more forward-looking the Central Bank is), the smaller the critical interest rate response coefficient ($\rho_\pi = \rho^H$) that is required for the presence of non-convergent cyclical fluctuations. Note that to the extent that central banks in practice do not respond too aggressively to inflation, these previous results imply that the more open the economy is or the more forward-looking the bank is, the more likely is that the specified rule will induced the aforementioned cyclical dynamics.

31 In the Appendix we state the Hopf Bifurcation Theorem.
Notwithstanding the relevance of these analytical results, it is crucial to investigate their quantitative importance. To accomplish this we borrow some values of the parameters from previous studies about small open economies and try to simulate the dynamics of the economy in the presence of a Hopf bifurcation. Following Schmitt-Grohé and Uribe’s (2001) study about Mexico we set: \( \theta_N = 0.36, \theta_T = 0.26, \phi = -11, \gamma = 175, \eta = 10^{-5}, r^* = 0.065 \) per year, \( \hat{\pi} = 0.0628 \) per year. We set \( \alpha = 0.44 \), that corresponds roughly to the imports to GDP share in Mexico during the 90’s. Finally we set \( k = 1000 \) and \( \rho_\pi = 2.18 \) in order to obtain cyclical fluctuations due to Hopf bifurcations with a period close to 4 years.\(^{32}\) Finally we assume that agents at time zero know \( b_0 = \bar{b} \) and expect an increase in consumption of non-traded goods of 3%. We summarize the parametrization in the following table.

<table>
<thead>
<tr>
<th>( \theta_N )</th>
<th>( \theta_T )</th>
<th>( \phi )</th>
<th>( \gamma )</th>
<th>( \eta )</th>
<th>( r^* )</th>
<th>( \alpha )</th>
<th>( \hat{\pi} )</th>
<th>( k )</th>
<th>( \bar{b} )</th>
<th>( \rho_\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.36</td>
<td>0.26</td>
<td>-11</td>
<td>175</td>
<td>( 10^{-5} )</td>
<td>0.065</td>
<td>0.44</td>
<td>0.0628</td>
<td>1000</td>
<td>0.7442</td>
<td>2.18</td>
</tr>
</tbody>
</table>

The results of this quantitative exercise are presented in Figure 3. As can be observed, this figure confirms the results in Proposition 5. The rule under analysis induces global cyclical fluctuations that never converge to the steady state. This type of dynamics is clear for the non-traded consumption, the real exchange rate, the nominal interest rate, the non-traded inflation, the traded inflation and the CPI-inflation. However the agents in this economy smooth completely consumption of traded goods and the current account stays balanced. This result is just a consequence of the fact that in the system (??) the interest rate rule does not affect neither the Euler equation of the consumption of traded goods nor the current account equation.

\(^{32}\)k and \( \rho_\pi \) are the parameters that specify the rule. Remember that the values of k and \( \rho_\pi \) are related in the sense that a lower value of k requires a lower value of \( \rho_\pi \) in order to induce cyclical fluctuations. However the lower k and \( \rho_\pi \) the longer the period of these oscillations. In these set-up \( k = 1000 \) implies that the central bank assigns a very high weight to the recent future inflation rates in the rule. In fact the rule becomes almost “contemporaneous”
the measure of inflation in the forward-looking rule may avoid the possibility of non-convergent cyclical
fluctuations due to Hopf bifurcations. The reason is that our analysis from equations (??) and (??) suggests,
we may loosely interpret $\alpha$ as the weight $w$ that the Central bank puts on the (weighted average of the
expected future) traded (flexible-price) inflation rate in the forward-looking rule. Hence from (??) it is
clear that as $w \to 0$ then the government responds exclusively to the (weighted average of the expected
future) non-traded (sticky-price) good inflation rate. Moreover if this is the case Figure 1 suggests that the
aforementioned cyclical fluctuations are still possible. The following Proposition formalizes this intuition.

**Proposition 6** Let $\Omega$ and $\rho^{HN}$ be defined as in Lemma 2 and assume that the central bank follows a forward-
looking rule such that $R = \bar{R} + \rho_\pi(\pi_{Nf} - \bar{\pi})$ with $\rho_\pi > 0$ and $\pi_{Nf} = k \int_t^{\infty} \pi_N(s)e^{-k(s-t)}ds$ with $k > 0$.
Assume that $\Omega < 1$ then there exists a critical value $\rho_\pi = \rho^{NH}$ such that the dynamic system described by
the rule and equations (??), (??), (??) and (??) displays a Hopf bifurcation.

**Proof.** See Appendix. ■

Proposition 6 shows that responding to the weighted average of expected future sticky-price (non-traded)
inflation rates may still induce non-convergent cyclical fluctuations in the economy. Once more the critical
value for the existence of this dynamics depends on the interest rate response coefficient to inflation $\rho_\pi =
\rho^{NH}$. However from Lemma 2 we know that this critical value is in fact related to structural parameters of
the economy such as the share of traded goods, $\alpha$, and the degrees of monopolistic competition, $\phi$, and price
stickiness, $\gamma$, in the non-traded sector, as well as to the weight that the monetary authority puts on expected
future sticky-price inflation rates in the rule, $k$. Although the relationship between this critical value of $\rho_\pi$ and $k$ is the same that we found for rules that responded to the CPI-inflation, the relationship between $\rho_\pi$ and $\alpha$ is the opposite. In other words, as shown by Figure 2 the smaller $\alpha$ is (the less open the economy is),
the smaller the critical interest rate response coefficient ($\rho_\pi = \rho^H$) that is necessary for the appearance of
cyclical fluctuations due to Hopf bifurcations.

It is simple to show that if the measure of inflation in the forward-looking rule corresponds to the flexible-
price (traded) inflation then our representation of the open economy does not display cyclical fluctuations.
This result in tandem with Proposition 6 suggest that from a global perspective it is not necessarily true
that the government should respond to the sticky-price (non-traded) inflation. After all, doing so is what
opens the possibility of aggregate instability due to Hopf-bifurcation cyclical fluctuations. Therefore this result challenges the following two proposals. The first one coincides with the results from the local analysis that suggest that in order to avoid (local) aggregate instability the government should respond to (target) the sticky-price (non-traded) inflation rate. This view has been supported by studies of optimal monetary policy in open economies such as Clarida et al. (2001), Devereux and Lane (2003) and Kollmann (2002). The second one is associated with the idea that it is important to target a modified inflation index that filters out the transitory effects of exchange rate movements as suggested by Ball (1999). In our model to the extent that there is a perfect degree of exchange rate pass through, the flexible-price or traded inflation coincides with the nominal depreciation rate. However our global equilibrium analysis suggests that responding to the sticky-price (non-traded) inflation (that is not affected by the swings of the nominal exchange rate) is what makes the economy more prone to aggregate instability.

5 Conclusions

In this paper we have studied the conditions under which (forward-looking) interest rate rules induce cyclical endogenous fluctuations and therefore aggregate macroeconomic instability in open economies. We find that these conditions depend not only on the specification of the rule but also on the degree of openness of the economy and on the measure of inflation to which the rule responds. Contrary to the results based on local equilibrium analysis, our global analysis warns policy makers about the possible perils of responding to the sticky-price (non-traded) inflation in the rule. It is precisely this type of response what opens the possibility of cycles due to a Hopf bifurcation in the open economy.

We are in the process of introducing imperfect exchange rate pass-through in the model to evaluate how our results may change. In addition we are studying the “super-criticality” properties of the cycles obtained in our model. To some extent it is crucial to determine if the cycles in our model are attractive.

Finally we want to mention that our results using a continuous time model are not subject to the critique raised by Carlstrom and Fuerst (2003). They have advocated the use of discrete time cash-in-advance models to pursue determinacy of equilibrium analysis since this set-up may capture the crucial role of the timing
of money transactions in the analysis. However our model although in continuous time has the flavor of a cash in advance model since the interest rate rule is a forward-looking rule defined in terms of the weighted average of expected future inflation rates.

6 Appendix

We transcribe without a proof the Hopf Bifurcation Theorem from Guckenheimer and Holmes (1983)

The Hopf Bifurcation Theorem

Suppose that the system \( \dot{x} = f(x) \), \( x \in \mathbb{R}^n \), \( \mu \in \mathbb{R} \) has an equilibrium \((x_0, \mu_0)\) at which the following properties are satisfied:

(H1) \( D_x f_{\mu_0}(x_0) \) has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. Then (H1) implies that there is a smooth curve of equilibria \((x(\mu), \mu)\) with \( x(\mu_0) = \mu_0 \). The eigenvalues \( \lambda(\mu), \bar{\lambda}(\mu) \) of \( D_x f_{\mu_0}(x(\mu)) \) which are imaginary at \( \mu = \mu_0 \) vary smoothly with \( \mu \). If, moreover,

(H2) \[
\frac{d}{d\mu}(\text{Re}\lambda(\mu))|_{\mu=\mu_0} = d \neq 0,
\]

then there is a unique three-dimensional center manifold passing through \((x_0, \mu_0)\) in \( \mathbb{R}^n \times \mathbb{R} \) and a smooth system of coordinates (preserving the planes \( \mu = \text{cons.} \)) for which the Taylor expansion of degree 3 on the center manifold is given by

\[
\begin{align*}
\dot{x} &= (dp + a(x^2 + y^2)) x - (\omega + c\mu + b(x^2 + y^2)) y \\
\dot{y} &= (\omega + c\mu + b(x^2 + y^2)) x - (d\mu + a(x^2 + y^2)) y
\end{align*}
\]

If \( a \neq 0 \) there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of \( \lambda(\mu_0), \bar{\lambda}(\mu_0) \) agreeing to second order with the paraboloid \( \mu = -\left(\frac{a}{\omega}\right)(x^2 + y^2) \). If \( a < 0 \) then these periodic solutions are stable limit cycles, while if \( a > 0 \), the periodic solutions are repelling.

Lemma 0

The matrix \( \Lambda \) (and therefore its characteristic polynomial \( \mathcal{P}(v) = 0 \)) in (??) has one negative eigenvalue (root) and one positive eigenvalue (root). That is \( v_1 > 0 \) and \( v_2 < 0 \).

Proof. Using the definition of \( \Lambda \) we obtain that \( \text{Det}(\Lambda) = -\eta \Delta_{21} < 0 \). But since \( \text{Det}(\Lambda) = v_1 v_2 \) and \( \text{Det}(\Lambda) = -\eta \Delta_{21} < 0 \) then we now that the two roots are real and they have opposite signs. With out loss of generality we can assume \( v_1 > 0 \) and \( v_2 < 0 \).
Proof for Lemma 1

Proof. We prove a) by contradiction. Assume that \( \rho^H > 1 \). Then by definition of \( \rho^H \) we have that 
\[
\frac{1}{\alpha} + \frac{\Omega(1-\alpha)^2}{2\alpha^2} + \frac{r^*}{2\alpha k} - \frac{\sqrt{\Phi}}{2\alpha r^k} \geq 1 \text{ if and only if } 2\alpha k + \Omega (1-\alpha)^2 + \alpha r^* - 2\alpha^2 k \geq \sqrt{\Phi}.
\]
Taking squares to both sides of the last inequality and using the definition of \( \Phi \) we have that \( \rho^H > 1 \) if and only if 
\[
0 \leq 4\Omega k (1-\alpha) [(1-\alpha) k + \alpha r^*] - 4\alpha k^2 (1-\alpha)^3 \Omega - 4\alpha^2 r^* k (1-\alpha) - 4\alpha^2 k^2 (1-\alpha)^2.
\]
And after some algebra we obtain that the last inequality holds if and only if 
\[
0 \leq -4\alpha^2 k (1-\alpha) (1-\Omega) [k (1-\alpha) + r^*],
\]
which is a contradiction since \( \alpha \in (0,1) \), \( \Omega < 1 \) and \( r^*, k > 0 \).

On the other hand assume that \( \rho^H \geq \frac{1}{\alpha} - \frac{(1-\alpha)\Omega}{\alpha} \). Then by definition of \( \rho^H \) we have that the last inequality holds if and only if 
\[
\left( \frac{\sqrt{\Phi}}{2\alpha r^k} \right)^2 \leq \left[ \frac{\Omega (1-\alpha)^2}{2\alpha^2} + \frac{r^*}{2\alpha k} - \frac{(1-\alpha)\Omega}{\alpha} \right]^2.
\]
After some algebra we derive that this holds if and only if 
\[
0 \leq \frac{(1-\alpha)^2 (\Omega (\Omega - 1))}{\alpha^2},
\]
which is a contradiction given that \( \alpha \in (0,1) \) and \( 0 < \Omega < 1 \). Moreover note that since \( \frac{1}{\alpha} - \frac{(1-\alpha)\Omega}{\alpha} < \frac{1}{\alpha} \) and \( \rho^H < \frac{1}{\alpha} - \frac{(1-\alpha)\Omega}{\alpha} \) then it follows that \( \rho^H < \frac{1}{\alpha} \).

To prove b) we rewrite \( \rho^H \) as
\[
\rho^H = \frac{[\Omega (1-\alpha)^2 + 2\alpha k + \alpha r^*] - \sqrt{\Omega (1-\alpha)^2 + 2\alpha k + \alpha r^*}^2 - 4\alpha^2 k [r^* + k - \Omega (1-\alpha) r^*]}{2\alpha k^2}
\]
the first part of b), \( \lim_{\alpha \to 1} \rho^H = 1 \), is trivial. For the second part we need to apply L'Hôpital’s rule twice to obtain \( \lim_{\alpha \to 0} \rho^H = \frac{1}{\Omega} + \frac{r^* (1-\alpha)}{2\alpha k} > 1 \), where the last inequality follows since \( \Omega < 1 \) and \( r^*, k > 0 \).

To prove c) we start by observing that the expression of \( \rho^H \) in (??) corresponds to one of the roots of the quadratic equation \( P(\rho_x) = k \alpha^2 p^2_x - [\Omega (1-\alpha)^2 + 2\alpha k + \alpha r^*] \rho_x + [r^* + k - \Omega (1-\alpha) r^*] = 0 \). In particular, we are interested in the root that satisfies \( \rho < \frac{1}{\alpha} \). We know that \( \rho^H \) satisfies this condition since 
\[
\rho^H < \frac{[\Omega (1-\alpha)^2 + 2\alpha k + \alpha r^*] - \sqrt{[\Omega (1-\alpha)^2 + 2\alpha k + \alpha r^*]^2}}{2\alpha k^2} = \frac{1}{\alpha}.
\]
In this case using part a) of this Lemma, and the assumptions that \( \Omega < 1, \alpha \in (0,1) \) and \( r^*, k > 0 \) we have by the Implicit Function Theorem (IFT) that
\[
\frac{\partial \rho^H}{\partial \alpha} = -\frac{\partial P(\rho_x)}{\partial \alpha} = -\frac{2 \alpha k \rho^H [\rho^H - \frac{(1-\alpha)\Omega}{\alpha}] + r^* (\rho^H - \Omega)}{2 \alpha k (1-\alpha \rho^H) + \alpha r^* + \Omega k (1-\alpha)^2} < 0
\]
\[
\frac{\partial \rho^H}{\partial k} = -\frac{\partial P(\rho_x)}{\partial k} = \frac{(\alpha \rho^H - 1)^2 + \frac{(1-\alpha)\Omega r^*}{k}}{2 \alpha k (1-\alpha \rho^H) + \alpha r^* + \Omega k (1-\alpha)^2} > 0
\]
\[
\frac{\partial \rho^H}{\partial \phi} = \frac{\partial P(\rho_x)}{\partial \phi} = \frac{k (1-\alpha)^2 \rho^H + (1-\alpha) r^*}{\phi N r^* k^2 [2 \alpha k (1-\alpha \rho^H) + \alpha r^* + \Omega k (1-\alpha)^2]} > 0
\]
\[
\frac{\partial \rho^H}{\partial \gamma} = \frac{\partial P(\rho_x)}{\partial \gamma} = \frac{k (1-\alpha)^2 \rho^H (1-\alpha) r^*}{\gamma^2 [2 \alpha k (1-\alpha \rho^H) + \alpha r^* + \Omega k (1-\alpha)^2]} > 0
\]

Proof for Proposition 1
Proof. First we make the following transformations \(x_T = \ln c_T - \ln \bar{c}_T, x_N = \ln c_N - \ln \bar{c}_N, d = b - \bar{b}, q_f = \pi_f - \bar{\pi} \) and \(q_N = \pi_N - \bar{\pi}\). Using these transformations in tandem with (??) we can derive the following linearized system whose steady state corresponds to \((\bar{x}_T, \bar{x}_N, \bar{d}, \bar{q}_f, \bar{q}_N) = (0, 0, 0, 0, 0)\)

\[
\begin{pmatrix}
\dot{x}_T \\
\dot{d} \\
\dot{q}_f \\
\dot{q}_N \\
\dot{x}_N
\end{pmatrix} = \begin{pmatrix}
0 & \eta & 0 & 0 & 0 \\
\Delta_{21} & r^* & 0 & 0 & 0 \\
0 & 0 & k(1 - \alpha \rho) & -k(1 - \alpha) & 0 \\
0 & 0 & 0 & r^* & \Delta_{45} \\
0 & 0 & \rho_\pi & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x_T \\
d \\
q_f \\
q_N \\
x_N
\end{pmatrix}
\]

(36)

where \(\Delta_{21} = \bar{c}_T \left[ 1 + \left( \frac{\theta_T}{1 - \theta_T} \right) \left( \frac{1}{e_{PT}} \frac{1}{\bar{c}_T} \right) \right] > 0\) and \(\Delta_{45} = \frac{(1 + \phi)(1 - \alpha)}{\gamma_{PN}} < 0\). It is important to observe that the matrix \(\Delta\) has a block triangular structure then we can partition the system in the following two

\[
\begin{pmatrix}
\dot{x}_T \\
\dot{d}
\end{pmatrix} = \begin{pmatrix}
0 & \eta \\
\Delta_{21} & r^*
\end{pmatrix}
\begin{pmatrix}
x_T \\
d
\end{pmatrix}
\]

(37)

and

\[
\begin{pmatrix}
\dot{q}_f \\
\dot{q}_N \\
\dot{x}_N
\end{pmatrix} = \begin{pmatrix}
k(1 - \alpha \rho) & -k(1 - \alpha) & 0 \\
0 & r^* & \Delta_{45} \\
\rho_\pi & -1 & 0
\end{pmatrix}
\begin{pmatrix}
q_f \\
q_N \\
x_N
\end{pmatrix}
\]

(38)

These previous analysis allows to decompose the characteristic polynomial, \(P(v) = 0\), associated to \(\Delta\) in (??) as

\[
P(v) = \left[ v^2 - \text{Trace}(\Lambda)v + \text{Det}(\Lambda) \right] \left[ v^3 - \text{Trace}(\Sigma)v^2 + S(\Sigma)v - \text{Det}(\Sigma) \right] = 0
\]

(39)

where \(\text{Trace}(\Sigma), \text{Det}(\Sigma)\), and \(S(\Sigma)\) denote the trace, the determinant and the sum of the 2 \times 2 principal minors of the matrix \(\Sigma\). In addition using the definition of \(\Omega\) and \(\Sigma\) we obtain

\[
\text{Trace}(\Sigma) = r^* + k(1 - \alpha \rho_\pi)
\]

(40)

\[
S(\Sigma) = kr^* \alpha \left( \frac{1}{\alpha} - \Omega(1 - \alpha) \right) - \rho_\pi
\]

(41)

\[
\text{Det}(\Sigma) = \Delta_{45} k(1 - \rho_\pi)
\]

(42)
The determinacy of equilibrium analysis requires to determine the sign of each of the five \((i = 1, 2, 3, 4, 5)\) roots \((v_i)\) of the characteristic polynomial \(P(v) = 0\) in (7). In other words we need to determine the sign of the two \((i = 1, 2)\) roots \((v_i)\) of the characteristic polynomial \(P_1(v) = 0\) and the sign of the three \((i = 3, 4, 5)\) roots \((v_i)\) of the characteristic polynomial \(P_2(v) = 0\).

By Lemma 0 in this Appendix we know that both of the eigenvalues of \(\Lambda\) (roots of \(P_1(v) = 0\)) are real and that they have opposite signs. That is without loss of generality we know that \(v_1 > 0\) and \(v_2 < 0\).

We continue characterizing the roots of \(P_2(v) = 0\) or eigenvalues of \(\Sigma\), but this leads us to prove parts a), b) and c) of the Proposition.

By theorem 1.2.12 from Horn and Johnson (1985) we know that \(\text{Trace}(\Sigma) = v_3 + v_4 + v_5\), \(S(\Sigma) = v_3v_4 + v_3v_5 + v_4v_5\) and \(\text{Det}(\Sigma) = v_3v_4v_5\) where the \(v_i\)’s for \(i = 3, 4, 5\) correspond to the eigenvalues for \(\Sigma\) or equivalently to the roots of \(P_2(v) = 0\) in (7).

To prove a) first notice that since \(\rho_\pi < 1 < \frac{1}{\alpha}\) then using (7) and (8) we can infer that \(\text{Trace}(\Sigma) > 0\) and \(\text{Det}(\Sigma) < 0\). Using this result and Theorem 1.2.12 from Horn and Johnson (1985) we can infer that \(\Sigma\) has one eigenvalue (root) with a negative real part and two eigenvalues (roots) with positive real parts. This in tandem with (7) implies that \(\Delta (P(v) = 0)\), has two eigenvalues (roots) with negative real parts and three eigenvalues (roots) with positive real parts. Given that there are four jump variables \((x_N, q_f, q_N, x_N)\), the number of jump variables is greater than the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a continuum (multiple) of equilibria.

To prove b) we use the Theorem by Routh-Hurwicz\(^{33}\) that states that the number of roots of \(\Sigma\) with positive real parts is equal to the number of variations of sign in the scheme

\[
1 - \text{Trace}(\Sigma) \quad \frac{S(\Sigma)\text{Trace}(\Sigma) - \text{Det}(\Sigma)}{\text{Trace}(\Sigma)} \quad - \text{Det}(\Sigma)
\]

By assumption we have \(1 < \rho_\pi < \rho^H\) and by Lemma 1 we know that \(\rho^H < \frac{1}{\alpha}\). Then we can deduce that \(1 < \rho_\pi < \frac{1}{\alpha}\). This result together with (7) and (8) allow us to infer that \(\text{Trace}(\Sigma) > 0\) and \(\text{Det}(\Sigma) > 0\). Moreover note that since \(\text{Trace}(\Sigma) > 0\) then \(\text{sign} \left[ \frac{S(\Sigma)\text{Trace}(\Sigma) - \text{Det}(\Sigma)}{\text{Trace}(\Sigma)} \right] = \text{sign} \left[ S(\Sigma)\text{Trace}(\Sigma) - \text{Det}(\Sigma) \right] \).

Using (7), (8) and (9) we have that \(S(\Sigma)\text{Trace}(\Sigma) - \text{Det}(\Sigma) = 0\) if and only if

\[
E(\rho_\pi) = k\alpha^2\rho_\pi^2 - \left[ \Omega k(1 - \alpha)^2 + 2ak + \alpha r^* \right] \rho_\pi + \left[ r^* + k - \Omega(1 - \alpha)r^* \right] = 0
\]

Clearly this is a quadratic equation in terms of \(\rho_\pi\) and there are two roots \(\rho_\pi^{1,2}\). But note that \(\rho^H\) is one of the roots of this quadratic equation. In fact we can rewrite \(\rho^H\) as we did in (6) in the Proof for Lemma 1. More importantly \(\rho^H\) is the root that we are interested in. The reason is that as argued before we know that \(1 < \rho_n < \frac{1}{\alpha}\) and for the other root, say \(\rho_\pi^2\), it is simple to show that \(\rho_\pi^2 > \frac{1}{\alpha}\). However we will characterize the equilibrium for either \(\frac{1}{\alpha} < \rho_\pi^2 < \rho_\pi\) or \(\frac{1}{\alpha} < \rho_\pi < \rho_\pi^2\) in the proof for part c).

\(^{33}\)See Gantmacher (1960).
It is simple to verify that for a small $\varepsilon$, if $\rho_\pi = \rho^H - \varepsilon$ then $E(\rho_\pi) > 0$ and if $\rho_\pi = \rho^H + \varepsilon$ then $E(\rho_\pi) < 0$. Therefore using this and our previous analysis we can conclude that if $\rho_\pi = \rho^H - \varepsilon$ then $S(\Sigma) = \text{Trace}(\Sigma) - \text{Det}(\Sigma) > 0$ and if $\rho_\pi = \rho^H + \varepsilon$ then $S(\Sigma) = \text{Trace}(\Sigma) - \text{Det}(\Sigma) < 0$.

Therefore we have that if $1 < \rho_\pi < \rho^H$ then $\text{Trace}(\Sigma) > 0$, $S(\Sigma) = \text{Trace}(\Sigma) - \text{Det}(\Sigma) > 0$ and $\text{Det}(\Sigma) > 0$. Applying the Theorem by Routh-Hurwicz we may infer that $\Sigma$ is stable with positive real parts. This in tandem with our previous proof of part b) implies that for a small $\varepsilon$ we have that $\rho_\pi = \rho^H + \varepsilon < \frac{1-\Omega}{\alpha} + \Omega$ and $\frac{1-\Omega}{\alpha} + \Omega < \rho_\pi$. For the first case, $\rho^H < \rho_\pi < \frac{1-\Omega}{\alpha} + \Omega$, our previous proof of part b) implies that for a small $\varepsilon$ we have that $\rho_\pi = \rho^H + \varepsilon < \frac{1-\Omega}{\alpha} + \Omega$ then $S(\Sigma) = \text{Trace}(\Sigma) - \text{Det}(\Sigma) < 0$. Moreover since $\rho^H < \rho_\pi < \frac{1-\Omega}{\alpha} + \Omega$, $\frac{1-\Omega}{\alpha} + \Omega < \frac{1}{\alpha}$ and by Lemma 1, $1 < \rho^H$, then we may infer that $1 < \rho_\pi < \frac{1}{\alpha}$. But this inequality in tandem with (10) allows us to infer that $\text{Trace}(\Sigma) > 0$ and $\text{Det}(\Sigma) > 0$. Therefore for $\rho^H < \rho_\pi < \frac{1-\Omega}{\alpha} + \Omega$ we have that $\text{Trace}(\Sigma) > 0$, $S(\Sigma) = \text{Trace}(\Sigma) - \text{Det}(\Sigma) < 0$ and $\text{Det}(\Sigma) > 0$. Applying the Theorem by Routh-Hurwicz we may infer that $\Sigma$ is stable with positive real parts and two eigenvalues (roots) with negative real parts.

For the second case, $\frac{1-\Omega}{\alpha} + \Omega < \rho_\pi$, we can use (10) to conclude that $S(\Sigma) < 0$. Moreover since $1 < \frac{1-\Omega}{\alpha} + \Omega$ then $1 < \rho_\pi$ which together with (10) imply that $\text{Det}(\Sigma) > 0$. Since $S(\Sigma) < 0$ and $\text{Det}(\Sigma) > 0$ then we can infer by theorem 1.2.12 from Horn and Johnson (1985) that $\Sigma$ is stable with positive real parts and two eigenvalues (roots) with negative real parts.

Therefore for both cases $\rho^H < \rho_\pi < \frac{1-\Omega}{\alpha} + \Omega$ and $\frac{1-\Omega}{\alpha} + \Omega < \rho_\pi$ we have that $\Sigma$ is stable with positive real parts and two eigenvalues (roots) with negative real parts. This in tandem with $v_1 > 0$ and $v_2 < 0$ imply that the matrix $\Delta$ ($P(v) = 0$) has three eigenvalues (roots) with negative real parts and two eigenvalues (roots) with positive real parts. Given that there are four jump variables $(x_N, q_f, q_N, x_N)$, the number of jump variables is greater than the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a continuum (multiple) of equilibria.

**Proof for Proposition 2**

**Proof.** We concentrate on the system (10) from the Proof of Proposition 1. Hence we focus on characterizing the eigenvalues of the matrix $\Delta$ or equivalently on the roots of $P(v) = 0$ in (10). Note that $\Lambda$ and $\Sigma$ were defined in the Proof of Proposition 1. Moreover from Lemma 0 we know that both eigenvalues of $\Lambda$ (roots of $P_1(v) = 0$) are real and that they have opposite signs. That is without loss of generality we know that $v_1 > 0$ and $v_2 < 0$. 27
Second, we need to characterize the eigenvalues of $\Sigma$ or roots of $P_2(v) = 0$ defined in (??). To do so we consider two cases for $\rho_\pi: \rho_\pi < 1$ and $\rho_\pi > 1$. For the first case $\rho_\pi < 1$ the proof is identical to the proof of part a) of Proposition 1.

For the second case, $\rho_\pi > 1$, note that $\frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha} = \Omega + \frac{1-\Omega}{\alpha} < 1$ provided that $\Omega > 1$ and $\alpha \in (0, 1)$. Therefore this result and $\rho_\pi > 1$ imply that $\rho_\pi > \frac{1}{\alpha} - \frac{\Omega(1-\alpha)}{\alpha}$. Using this and (??) and (??) from the proof of Proposition 1, we can conclude that in this case $Det(\Sigma) > 0$ and $S(\Sigma) < 0$. Then by Theorem 1.2.12 from Horn and Johnson(1985) we can infer that $\Sigma (P_2(v) = 0)$ has one eigenvalue (root) with a positive real part and two eigenvalues (roots) with negative real parts. This in tandem with $v_1 > 0$ and $v_2 < 0$ imply that $\Delta (P(v) = 0)$, has three eigenvalues (roots) with negative real parts and two (roots) eigenvalues with positive real parts. Given that there are four jump variables $(x_N, q_f, q_N, x_N)$, the number of jump variables is greater than the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a continuum (multiple) of equilibria. ■

**Proof for Proposition 3**

**Proof.** First note that using equations $\dot{\epsilon}_f = k(\epsilon_f - \epsilon)$, (??), (??), (??), (??) and (??) in tandem with the transformations $x_T = \ln c_T - \ln \bar{c}_T$, $x_N = \ln c_N - \ln \bar{c}_N$, $d = b - \bar{b}$, $q_{ef} = \epsilon_f - \bar{\pi}$ and $q_N = \pi_N - \bar{\pi}$ can derive the following linearized system whose steady state corresponds to $(\bar{x}_T, \bar{x}_N, \bar{d}, \bar{q}_f, \bar{q}_N) = (0, 0, 0, 0, 0)$

\[
\begin{pmatrix}
\dot{x}_T \\
\dot{d} \\
\dot{q}_f \\
\dot{q}_N \\
\dot{x}_N \\
\end{pmatrix} =
\begin{pmatrix}
0 & \eta & 0 & 0 & 0 \\
\Delta_{21} & r^* & 0 & 0 & 0 \\
0 & k\eta & k(1 - \rho_\pi) & 0 & 0 \\
0 & 0 & 0 & r^* & \Delta_{45} \\
0 & 0 & \rho_\pi & -1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_T \\
\dot{d} \\
\dot{q}_f \\
\dot{q}_N \\
x_N \\
\end{pmatrix}
\]

(43)

In this case the characteristic polynomial, $P(v) = 0$, associated to $\Delta_\epsilon$ can be written as

\[
P(v) = \left[v^2 - \text{Trace}(\Lambda) v + \text{Det}(\Lambda)\right] \left[v^3 - \text{Trace}(\Sigma_1) v^2 + S(\Sigma_1) v - \text{Det}(\Sigma_1)\right] = 0
\]

(44)

where $\Lambda$ was defined in the (??), and

\[
\Sigma_1 =
\begin{pmatrix}
k(1 - \rho_\pi) & 0 & 0 \\
0 & r^* & \Delta_{45} \\
\rho_\pi & -1 & 0 \\
\end{pmatrix}
\]

and $\Delta_{45} = \frac{(1+\phi)(1-\alpha)}{\theta N} < 0$. The determinacy of equilibrium analysis requires to determine the sign of each of the five ($i = 1, 2, 3, 4, 5$) roots ($v_i$) of the characteristic polynomial $P(v) = 0$ in (??) that correspond to the
eigenvalues of the matrix $\Delta$. In other words we need to determine the sign of the two $(i = 1, 2)$ roots $(v_i)$ of the characteristic polynomial $\mathcal{P}_1(v) = 0$ and the sign of the three $(i = 3, 4, 5)$ roots $(v_i)$ of the characteristic polynomial $\mathcal{P}_2(v) = 0$. However from the proof of Proposition 1 we know that both of the eigenvalues of $\Lambda$ (roots of $\mathcal{P}_1(v) = 0$) are real and that they have opposite signs. That is without loss of generality we know that $v_1 > 0$ and $v_2 < 0$.

We proceed characterizing the eigenvalues (roots) of $\Sigma_1$ ($\mathcal{P}_2(v) = 0$). Since $\Sigma_1$ is block triangular we know that one of the eigenvalues corresponds to $v_3 = k(1 - \rho_\pi)$ and the other two correspond to the roots $v_4$ and $v_5$ of the characteristic equation $\mathcal{P}_2(v) = v^2 - r^* v + \Delta_{45}$. Then by theorem 1.2.12 from Horn and Johnson (1985) we can infer that $\Delta_{45} = v_4 v_5$. But this and the fact that $\Delta_{45} < 0$ imply that the roots $v_4$ and $v_5$ have opposite signs. Then with out loss of generality we have that $v_4 > 0$ and $v_5 < 0$.

Moreover if $\rho_\pi > 1$ then $v_3 < 0$ whereas if $\rho_\pi < 1$ then $v_3 > 0$. Then our analysis of the roots, $v_i$ with $i = 1, 2, 3, 4, 5$ implies that $\mathcal{P}(v) = 0$ in (56) has at most three positive roots. Given that there are four jump variables ($x_N, q_f, q_N, x_N$), the number of jump variables is greater than the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a continuum (multiple) of equilibria. ■

**Proof for Lemma 2**

**Proof.** To prove a) we start by observing that since $\Omega < 1$ and $\alpha \in (0, 1)$ then $\Omega(1 - \alpha) < 1$. Using this, the definition of $\rho^{HN}$ and $r^*, k > 0$, the result in a) follows.

To prove b) and c) is straightforward. Take the limits and the derivatives of $\rho^{HN}$. Note that $\frac{1}{\Omega} + \frac{r^*(1 - \Omega)}{\Omega k} > 1$ follows from $\Omega < 1$ and $r^*, k > 0$. ■

**Proof for Proposition 4**

**Proof.** First note that using equations $\dot{\pi}_{Nf} = k(\pi_{Nf} - \pi_N)$, (??), (??), (??), (??), and (??) in tandem with the transformations $x_T = \ln c_T - \ln c_T$, $x_N = \ln c_N - \ln c_N$, $d = b - b$, $q_{NF} = \pi_{NF} - \pi$ and $q_N = \pi_N - \pi$ can derive the following linearized system whose steady state corresponds to $(\bar{\bar{x}}_T, \bar{x}_N, \bar{\bar{d}}, \bar{q}_f, \bar{q}_N) = (0,0,0,0,0)$

$$
\begin{pmatrix}
\dot{x}_T \\
\dot{d} \\
\dot{q}_{NF} \\
\dot{q}_N \\
\dot{\bar{\bar{x}}}_N
\end{pmatrix} =
\begin{pmatrix}
0 & \eta & 0 & 0 & 0 \\
\Delta_{21} & r^* & 0 & 0 & 0 \\
0 & 0 & k & -k & 0 \\
0 & 0 & 0 & r^* & \Delta_{45} \\
0 & 0 & \rho_\pi & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x_T \\
d \\
q_{NF} \\
q_N \\
x_N
\end{pmatrix}
$$

(45)

In this case the characteristic polynomial, $\mathcal{P}(v) = 0$, associated to $\Delta_N$ can be written as

$$
\mathcal{P}(v) = \left[ v^2 - \text{Trace}(\Lambda) v + \text{Det}(\Lambda) \right] \left[ v^3 - \text{Trace}(\Sigma_2) v^2 + S(\Sigma_2) v - \text{Det}(\Sigma_2) \right] = 0
$$

(46)
where \(\Delta\) was defined in the proof of Proposition 1, and

\[
\Sigma_2 = \begin{pmatrix} k & -k & 0 \\ 0 & r^* & \Delta_{45} \\ \rho_\pi & -1 & 0 \end{pmatrix}
\]  \hspace{1cm} (47)

and \(\Delta_{45} = \frac{(1+\phi)(1-\alpha)}{\theta_N} < 0\). The determinacy of equilibrium analysis requires to determine the sign of each of the five \((i = 1, 2, 3, 4, 5)\) roots \((v_i)\) of the characteristic polynomial \(P(v) = 0\) in (??) that correspond to the eigenvalues of the matrix \(\Delta\). In other words we need to determine the sign of the two \((i = 1, 2)\) roots \((v_i)\) of the characteristic polynomial \(P_1(v) = 0\) and the sign of the three \((i = 3, 4, 5)\) roots \((v_i)\) of the characteristic polynomial \(P_2(v) = 0\). However from the proof of Proposition 1 we know that both of the eigenvalues of \(\Lambda\) (roots of \(P_1(v) = 0\)) are real and that they have opposite signs. That is without loss of generality we know that \(v_1 > 0\) and \(v_2 < 0\).

We proceed characterizing the eigenvalues (roots) of \(\Sigma_2\) \((P_2(v) = 0)\). Using the definitions of \(\Omega\) and \(\Sigma_2\) we can derive its trace, its sum of the 2 \(2\) principal minors and its determinant as

\[
\text{Trace}(\Sigma_2) = r^* + k > 0
\]  \hspace{1cm} (48)

\[
S(\Sigma_2) = kr^*(1-\alpha) \left( \frac{1}{1-\alpha} - \Omega \right)
\]  \hspace{1cm} (49)

\[
\text{Det}(\Sigma_2) = \Delta_{45} k (1-\rho_\pi)
\]  \hspace{1cm} (50)

By theorem 1.2.12 from Horn and Johnson (1985) we know that \(\text{Trace}(\Sigma) = v_3 + v_4 + v_5\), \(S(\Sigma) = v_3v_4 + v_3v_5 + v_4v_5\) and \(\text{Det}(\Sigma) = v_3v_4v_5\) where the \(v_i\)’s for \(i = 3, 4, 5\) correspond to the eigenvalues for \(\Sigma\) or equivalently to the roots of \(P_2(v) = 0\) in (??).

We start by proving a). Since \(\rho_\pi < 1\) then \(\text{Det}(\Sigma_2) < 0\). Using this, the fact that \(\text{Trace}(\Sigma_2) > 0\) and the theorem 1.2.12 from Horn and Johnson (1985) we can infer that \(\Sigma_2\) \((P_2(v) = 0)\) has two eigenvalues (roots) with positive real parts and one eigenvalue (root) with a negative real part. This in tandem with \(v_1 > 0\) and \(v_2 < 0\) imply that \(\Delta\) \((P(v) = 0)\) has three eigenvalues (roots) with negative real parts and two eigenvalues (roots) with positive real parts. Given that there are four jump variables \((x_N, q_f, q_N, x_N)\), the number of jump variables is greater than the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a continuum (multiple) of equilibria.

To prove b) we use the Theorem by Routh-Hurwicz that we stated in the proof for Proposition 1. By assumption \(1 < \rho_\pi\) then \(\text{Det}(\Sigma_2) > 0\). Moreover we know that since \(\text{Trace}(\Sigma_2) > 0\) then \(\text{sign} \left[ \frac{S(\Sigma_2)\text{Trace}(\Sigma_2) - \text{Det}(\Sigma_2)}{\text{Trace}(\Sigma_2)} \right] = \text{sign} [S(\Sigma_2)\text{Trace}(\Sigma_2) - \text{Det}(\Sigma_2)]\). Using (??), (??)and (??) it is simple to show that \(S(\Sigma_2)\text{Trace}(\Sigma_2) - \text{Det}(\Sigma_2) \leq 0\) if and only if \(\rho_\pi \leq \rho_{HN}\). Note that by Lemma 2 we know that if \(\Omega < 1\) then \(1 < \rho_{HN}\). Therefore we know that the inequality \(1 < \rho_\pi < \rho_{HN}\) is well defined. If \(1 < \rho_\pi < \rho_{HN}\) then \(S(\Sigma_2)\text{Trace}(\Sigma_2) - \text{Det}(\Sigma_2) < 0\) and \(\text{Det}(\Sigma_2) > 0\). Using these and \(\text{Trace}(\Sigma_2) > 0\), we can apply the Theorem by Routh-Hurwicz to infer that \(\Sigma_2\) \((P_2(v) = 0)\) has three eigenvalues (roots) with positive real parts.
This in tandem with \( v_1 > 0 \) and \( v_2 < 0 \) imply that \( \Delta (P(v) = 0) \), has one eigenvalue (root) with a negative real parts and four eigenvalues (roots) with positive real parts. Given that there are four jump variables \((x_N, q_f, q_N, x_N)\), the number of jump variables is equal to the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a unique equilibrium.

To prove c) we proceed as we did to prove b). By Lemma 2 we know that \( \rho^{HN} > 1 \) given that \( \Omega < 1 \). Hence \( 1 < \rho^{HN} < \rho \) which implies that \( S(\Sigma)Trace(\Sigma) - Det(\Sigma) > 0 \) and \( Det(\Sigma) > 0 \). Using these and \( Trace(\Sigma) > 0 \), we can apply the Theorem by Routh-Hurwicz to infer that \( \Sigma \) with \( 0 < v < 2 < 0 \) imply that \( \Delta (P(v) = 0) \), has three eigenvalues (roots) with negative real parts and two eigenvalues (roots) with positive real parts. Given that there are four jump variables \((x_N, q_f, q_N, x_N)\), the number of jump variables is equal to the number of explosive eigenvalues. Applying the results of Blanchard and Kahn (1980) it follows that there is a continuum (multiple) equilibria.

**Proof for Proposition 5**

**Proof.** We focus on proving existence by checking that at \( \rho_H = \rho^H \) the sufficient conditions (H1) and (H2) of the Theorem of Hopf Bifurcations are satisfied. We can concentrate on the system \((\Sigma)\) and in particular on the matrix \( \Sigma \) of the system \((\Sigma)\). The reason is that by Lemma 0 we know that \( \Lambda \) has two non-zero real eigenvalues, \( v_1 > 0 \) and \( v_2 < 0 \).

To check (H1) we need to show that \( \Sigma \) has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. By theorem 1.2.12 from Horn and Johnson (1985) we know that \( Trace(\Sigma) = v_3 + v_4 + v_5, S(\Sigma) = v_3v_4 + v_3v_5 + v_4v_5 \) and \( Det(\Sigma) = v_3v_4v_5 \) where the \( v_i \)'s for \( i = 3, 4, 5 \) correspond to the eigenvalues for \( \Sigma \). Hence

\[
S(\Sigma)Trace(\Sigma) - Det(\Sigma) = -(v_3 + v_4)(v_4 + v_5)(v_3 + v_5) \quad (51)
\]

Moreover recall that in the proof of part b) in Proposition 1 we derived that \( S(\Sigma)Trace(\Sigma) - Det(\Sigma) = 0 \) if and only if \( \rho_\pi = \rho^H \). In addition using \((\Sigma)\) and part a) of Lemma 2 we can notice that if \( \rho_\pi = \rho^H \) then \( Det(\Sigma) > 0 \). Therefore we have that if \( \rho_\pi = \rho^H \) then \(-(v_3 + v_4)(v_4 + v_5)(v_3 + v_5) = 0 \) and \( v_3v_4v_5 > 0 \). This implies that if \( \rho_\pi = \rho^H \) then \( \Sigma \) has non zero eigenvalues with two possible alternatives: 1) all of them are real satisfying without loss of generality: \( v_3 = c > 0, v_4 = -c < 0 \) and \( v_5 \neq 0 \) or 2) there is a pair of pure imaginary eigenvalues and one real eigenvalue satisfying without loss of generality: \( v_3 = +bi, v_4 = -bi \) and \( v_5 \neq 0 \). Then we demonstrate that 1) is not feasible. The reason is that under this alternative \( S(\Sigma_2) = v_3v_4 + v_3v_5 + v_4v_5 = -c^2 < 0 \). However from \((\Sigma)\) and part a) of Lemma 1 we know that if \( \rho_\pi = \rho^H \) then \( S(\Sigma) > 0 \). Hence there is a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

To check (H2) we need to verify that \( \frac{d}{d\rho_\pi} (Re(v(\rho_\pi)))|_{\rho_\pi=\rho^H} \neq 0 \) where \( Re(v(\rho_\pi)) \) corresponds to the real part of the imaginary eigenvalues. Assume without loss of generality that \( v_3 = \hat{a} + \hat{c}i, v_4 = \hat{a} - \hat{b}i \) and
non-zero real eigenvalues, particular on the matrix $\Sigma (H2)$ of the Theorem of Hopf Bifurcations are satisfied. We can concentrate on the system \((\psi = \vec{\lambda} = 0\text{ if and only if } \Phi = \vec{\lambda} = 0\text{ and therefore }\)

Moreover from \((??), (??)\)and \((??)\) we have that

\[
S(\Sigma)\text{Trace}(\Sigma) - \text{Det}(\Sigma) = r^*k \{ \kappa a^2 \rho_x^2 - [\Omega k(1 - \alpha)^2 + 2ak + \alpha r^*] \rho_x + [r^* + k - \Omega(1 - \alpha)r^*] \}
\]

and therefore

\[
\frac{d}{d\rho_x} \left|_{\rho_x = \rho^H} \right. \left. S(\Sigma)\text{Trace}(\Sigma) - \text{Det}(\Sigma) \right| = -r^*k\sqrt{\Phi} \neq 0
\]

since \(\Phi = [\Omega(1 - \alpha)^2k + \alpha r^*]^2 + 4\Omega\kappa(1 - \alpha)[(1 - \alpha)k + \alpha r^*] > 0\) and \(r^*, k > 0\). Finally using \((??)\) and \((??)\) we can conclude that \( \frac{d\hat{a}}{d\rho_x} \left|_{\rho_x = \rho^H} \right. \neq 0\) .

**Proof for Proposition 6**

**Proof.** We focus on proving existence by checking that at \(\rho_x = \rho^HN\) the sufficient conditions \((H1)\) and \((H2)\) of the Theorem of Hopf Bifurcations are satisfied. We can concentrate on the system \((??)\) and in particular on the matrix \(\Sigma_2\) of the system \((??)\). The reason is that by Lemma 0 we know that \(\Lambda\) has two non-zero real eigenvalues, \(v_1 > 0\) and \(v_2 < 0\).

To check \((H1)\) we need to show that \(\Sigma_2\) has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. By theorem 1.2.12 from Horn and Johnson (1985) we know that \(\text{Trace}(\Sigma_2) = v_3 + v_4 + v_5\), \(\Sigma_2 = v_3v_4 + v_3v_5 + v_4v_5\) and \(\text{Det}(\Sigma_2) = v_3v_4v_5\) where the \(v_i\)'s for \(i = 3, 4, 5\) correspond to the eigenvalues for \(\Sigma_2\). Hence

\[
S(\Sigma_2)\text{Trace}(\Sigma_2) - \text{Det}(\Sigma_2) = -(v_3 + v_4)(v_4 + v_5)(v_3 + v_5).
\]

Moreover recall that in the proof of part b) in Proposition 1 we derived that \(S(\Sigma_2)\text{Trace}(\Sigma_2) - \text{Det}(\Sigma_2) = 0\) if and only if \(\rho_x = \rho^HN\). In addition using \((??)\) and part a) of Lemma 2 we can notice that if \(\rho_x = \rho^HN\) then \(\text{Det}(\Sigma_2) > 0\). Therefore we have that if \(\rho_x = \rho^HN\) then \((-v_3 + v_4)(v_4 + v_5)(v_3 + v_5) = 0\) and \(v_3v_4v_5 > 0\). This implies that if \(\rho_x = \rho^H\) then \(\Sigma\) has non zero eigenvalues with two possible alternatives: 1) all of them are real satisfying without loss of generality: \(v_3 = \hat{c} > 0\), \(v_4 = -\hat{c} < 0\) and \(v_5 \neq 0\) or 2) there is a pair of pure imaginary eigenvalues and one real eigenvalue satisfying without loss of generality: \(v_3 = +\hat{b}i\), \(v_4 = -\hat{b}i\) and \(v_5 \neq 0\). Then we demonstrate that 1) is not feasible. The reason is that under this alternative \(S(\Sigma_2) = v_3v_4 + v_3v_5 + v_4v_5 = -\hat{c}^2 < 0\). However from \((??)\) and \(\Omega < 1\) we know that \(S(\Sigma_2) > 0\). Hence there is a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

To check \((H2)\) we need to verify that \(\frac{d}{d\rho_x}(\text{Re}(v(\rho_x)))|_{\rho_x = \rho^HN} \neq 0\) where \(\text{Re}(v(\rho_x))\) corresponds to the real part of the imaginary eigenvalues. Assume without loss of generality that \(v_3 = \hat{a} + \hat{c}i\), \(v_4 = \hat{a} - \hat{b}i\) and \(v_5 = \hat{c}\).
Then condition (H2) reduces to \( \frac{d\hat{a}}{d\rho}\big|_{\rho_\pi=\rho^{HN}} \neq 0 \). From the definitions of the eigenvalues and (??) we derive that \( S(\Sigma_2)Trace(\Sigma_2) - Det(\Sigma_2) = -2\hat{a}\left(\hat{a}^2 + 2\hat{a}\hat{c} + \hat{b}^2 + \hat{c}^2\right) \). Taking the derivative of this expression with respect to \( \hat{a} \) and using the fact that \( \hat{a} = 0 \) at \( \rho_\pi = \rho^{HN} \) we obtain

\[
\frac{d\hat{a}}{d\rho_\pi}\big|_{\rho_\pi=\rho^{HN}} = -\frac{d[S(\Sigma_2)Trace(\Sigma_2) - Det(\Sigma_2)]}{d\rho_\pi}\big|_{\rho_\pi=\rho^{HN}} \frac{1}{2(\hat{b}^2 + \hat{c}^2)}
\]

Moreover from (??), (??)and (??) it is simple to show that

\[
\frac{d[S(\Sigma_2)Trace(\Sigma_2) - Det(\Sigma_2)]}{d\rho_\pi}\big|_{\rho_\pi=\rho^{HN}} = -r^*k^2\Omega \neq 0
\]

since \( \Omega, r^*, k > 0 \). Finally using (??) and (??) we can conclude that \( \frac{d\hat{a}}{d\rho_\pi}\big|_{\rho_\pi=\rho^{HN}} \neq 0 \). □

7 References


